Ehrhart Quasi-Polynomials of Almost Integral Polytopes

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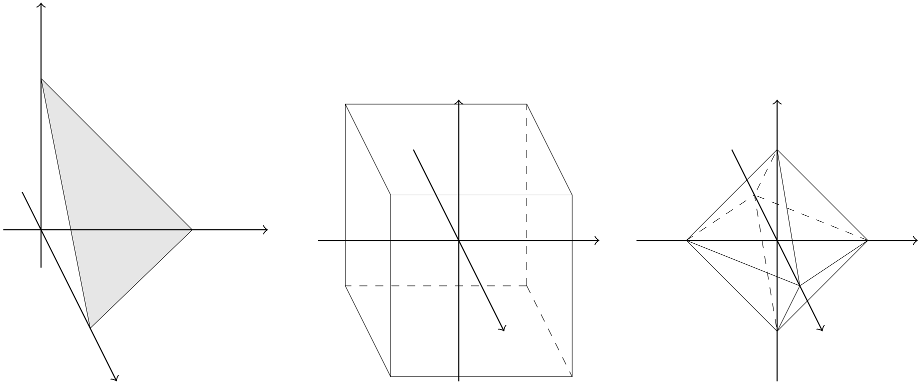
Map for today

- Preliminaries
- Translated Lattice Point Enumerator
- Characterizing Centrally Symmetric Polytopes
- Characterizing Zonotopes

First examples

Examples

standard d-simplex	d-cube	d-hypersimplex
$\triangle_d = \mathrm{conv}\{e_i\} \subset \mathbb{R}^{d+1}$	$C_d = \mathrm{conv}\{\pm e_i\}$	$\diamond_d = \mathrm{conv}\{e_i, -e_i\}$



(a) 2-simplex $\triangle_2$.

(b) 3-cube C_3.

(c) 3-hypersimplex $\diamond_3$.

Figure: First examples of polytopes

Minkowski sum

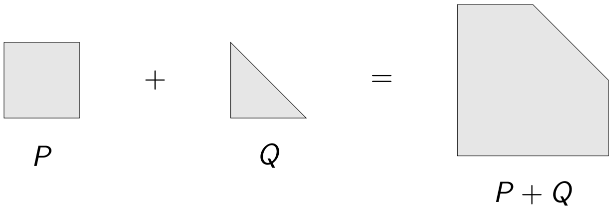


Figure: Minkowski sum of a square and a triangle

Paving a zonotope

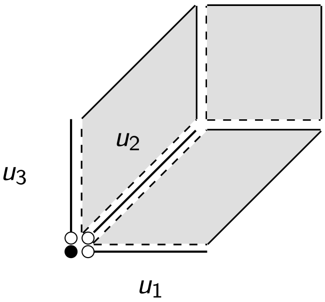
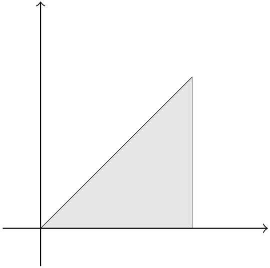


Figure: Paving a zonotope

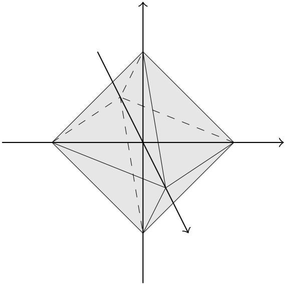
Motivating examples

Examples

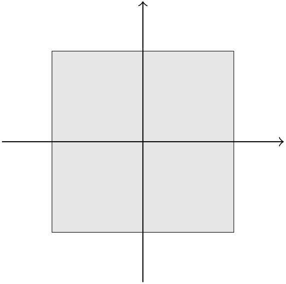
General polytope	Symmetric polytope	Zonotope
$\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2$	$\pm e_i \in \mathbb{R}^3, i = 1, 2, 3$	$\pm e_1 \pm e_2 \in \mathbb{R}^2$



(a) General polytope P.



(b) 3-hypersimplex $\diamond_3$.



(c) 2-cube C_2.

Zonotopes

Example Zonotope C_2

$$C_2 = \mathrm{conv}(\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}) \subset \mathbb{R}^2$$

$$c_3 = \begin{pmatrix}\frac{1}{5}\\\frac{1}{5}\end{pmatrix} \in \mathbb{R}^2$$

$$L_{c_3+C_2} = \begin{cases} (2t+1)^2 & \text{if } t \equiv 0 \bmod 5 \\ (2t)^2 & \text{if } t \equiv 1 \bmod 5 \\ (2t)^2 & \text{if } t \equiv 2 \bmod 5 \\ (2t)^2 & \text{if } t \equiv 3 \bmod 5 \\ (2t)^2 & \text{if } t \equiv 4 \bmod 5 \end{cases}$$

Symmetric polytopes

Example symmetric polytope $\diamond_3$

$$\diamond_3 = \mathrm{conv}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix} \right) \subset \mathbb{R}^3$$

$$c_2 = \begin{pmatrix}\frac{1}{5}\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix} \in \mathbb{R}^3$$

$$L_{c_2+\diamond_3}(t) = \begin{cases} \frac{4}{3}t^3 + 2t^2 + \frac{8}{3}t + 1 & \text{if } t \equiv 0 \bmod 5 \\ \frac{4}{3}t^3 - \frac{1}{3}t & \text{if } t \equiv 1 \bmod 5 \\ \frac{4}{3}t^3 - \frac{4}{3}t & \text{if } t \equiv 2 \bmod 5 \\ \frac{4}{3}t^3 - \frac{4}{3}t & \text{if } t \equiv 3 \bmod 5 \\ \frac{4}{3}t^3 - \frac{1}{3}t & \text{if } t \equiv 4 \bmod 5 \end{cases}$$

Symmetric polytopes

Example symmetric polytope $\diamond_3$

$$\diamond_3 = \text{conv}\left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\-1\end{pmatrix} \right) \subset \mathbb{R}^3$$

$$c_2 = \begin{pmatrix}\frac{1}{5}\\\frac{1}{5}\\\frac{1}{5}\end{pmatrix} \in \mathbb{R}^3$$

$$L_{c_2+\diamond_3}(t) = \begin{cases} \frac{4}{3}t^3 + 2t^2 + \frac{8}{3}t + 1 & \text{if } t \equiv 0 \bmod 5 \\ \frac{4}{3}t^3 - \frac{1}{3}t & \text{if } t \equiv 1 \bmod 5 \longleftarrow \\ \frac{4}{3}t^3 - \frac{4}{3}t & \text{if } t \equiv 2 \bmod 5 \longleftarrow \\ \frac{4}{3}t^3 - \frac{4}{3}t & \text{if } t \equiv 3 \bmod 5 \longleftarrow \\ \frac{4}{3}t^3 - \frac{1}{3}t & \text{if } t \equiv 4 \bmod 5 \longleftarrow \end{cases}$$

General polytope

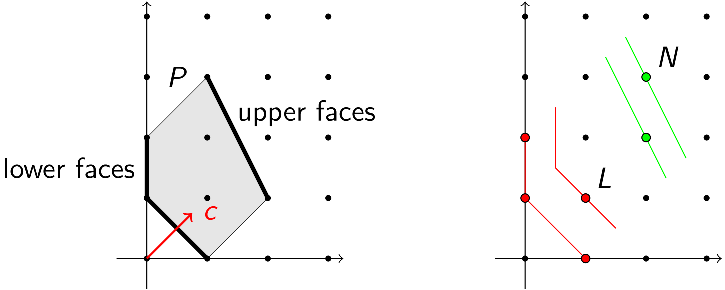
Example general polytope P

$$P = \text{conv}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right) \subset \mathbb{R}^2$$

$$c_1 = \begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\end{pmatrix} \in \mathbb{R}^2$$

$$L_{c_1+P}(t) = \begin{cases} \frac{1}{2}t^2 + \frac{3}{2}t + 1 & \text{if } t \equiv 0 \mod 3 \\ \frac{1}{2}t^2 - \frac{1}{2}t & \text{if } t \equiv 1 \mod 3 \\ \frac{1}{2}t^2 + \frac{1}{2}t & \text{if } t \equiv 2 \mod 3 \end{cases}$$

Upper and lower faces



(a) Upper and lower facets of P.

(b) The sets N and L for the polytope P.

Figure: New and lost points

(1) implies (2)

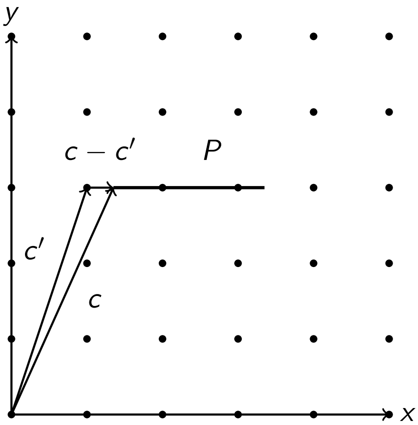


Figure: (1) implies (2)

Proof Characterizing centrally symmetric polytopes

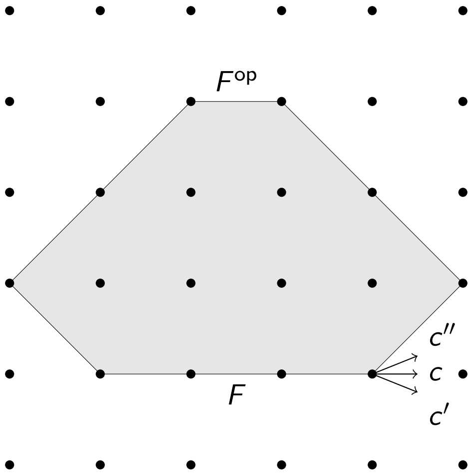
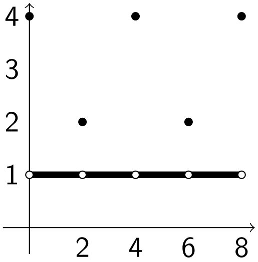


Figure: Translation by c, c' and c''.

Almost constant function

Let $P_1 = \text{conv}(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}) \subset \mathbb{R}^2$ and $c_1 = \begin{pmatrix}\frac{1}{2}\\\frac{1}{4}\end{pmatrix} \in \mathbb{Q}^2$. Then,

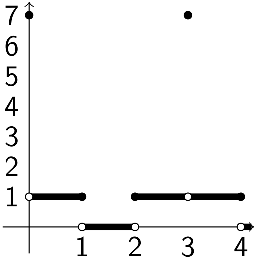
$$L_{c_1}^{P_1}(x) = \begin{cases} 4 & \text{if } x \in 4\mathbb{Z}_{\geq 0} \\ 2 & \text{if } x \in 2 + 4\mathbb{Z}_{\geq 0} \\ 1 & \text{else} \end{cases}$$



Not almost constant function

Let $P_2 = \text{conv}(\pm e_i | i = 1, 2, 3) \subset \mathbb{R}^d$ and $c_2 = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} \in \mathbb{Q}^3$. Then,

$$L_{c_2}^{P_2}(x) = \begin{cases} 7 & \text{if } x \in 3\mathbb{Z}_{\geq 0} \\ 1 & \text{if } x \in (k, k+1] \text{ for } k \in 3\mathbb{Z}_{\geq 0} \text{ or } x \in [k-1, k) \text{ for } k \in 3\mathbb{Z}_{>0} \\ 0 & \text{else} \end{cases}$$



Projection zonotopes

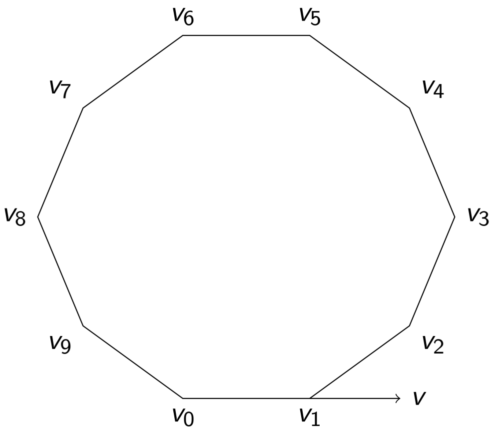


Figure: projection ($n = 10$)

THANK YOU FOR
YOUR ATTENTION!