Characteristic and Ehrhart quasi-polynomials for root systems

I, Introduction

1. Characteristic and Ehrhart quasi-polynomials
2. Free and supersolvable arrangements

II, Geometry and Enumeration on Weyl arrangements

1. Worpitzky-compatible arrangements
2. $\mathcal{A}$-Eulerian polynomials

III, Consideration on type A

1. Compatible arrangements and cocomparability graphs
2. Application to graph polynomials of cocomparability graphs

Main references

1. A.U. Ashraf, T.N. Tran, M. Yoshinaga. Eulerian polynomials for subarrangements of Weyl arrangements
Adv. Appl. Math.
2. T.N. Tran and A. Tsuchiya. Worpitzky-compatible subarr. of braid arr. and cocomp. graphs arxiv. 2007.01248.

I, Introduction

1. Characteristic and Ehrhart quasi-pol.

Def (Quasi-pol.)

- A function $g : \mathbb{Z} \to \mathbb{C}$ is called a quasi-polynomial if $\exists \rho \in \mathbb{Z}_{>0}$ (period) and $f^k(t) \in \mathbb{Q}[t]$ $(1 \leq k \leq \rho)$ (k-constituent) s.t.

$$g(q) = \begin{cases} f^1(q) & q \equiv 1 \mod \rho \\ f^2(q) & q \equiv 2 \mod \rho \\ \vdots & \\ f^\rho(q) & q \equiv \rho \mod \rho \end{cases}$$

- Equivalently,

$$g(q) = c_d(q) q^d + c_{d-1}(q) q^{d-1} + \cdots + c_0(q)$$

$c_i : \mathbb{Z} \to \mathbb{Q}$ is a periodic function with integral period i.e., $c_i(q) = c_i(q + T_i)$

The above ρ can be chosen as $\rho = \mathrm{lcm}(T_1, \ldots, T_d)$

- The smallest ρ is called the minimum period of g

Thm (Ehrhart 1962)

let $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \simeq \mathbb{Z}^{\ell}$ a lattice in $\mathbb{R}^{\ell}$.

let P be a polytope with vertices in $\bigoplus_{i=1}^{\ell} \mathbb{Q}\alpha_i$.

If $q \in \mathbb{Z}_{>0}$, then

$$L_P(q) := \#(qP \cap \Gamma)$$

is a quasi-pol. in q of degree $\dim(P)$ with coefficients in $\mathbb{Q}$.

$L_P(q)$ is called the Ehrhart quasi-pol of P w.r.t. the lattice Γ.

- The denominator $D(P)$ of P is the smallest $k \in \mathbb{Z}_{>0}$ for which the vertices of kP are in Γ.

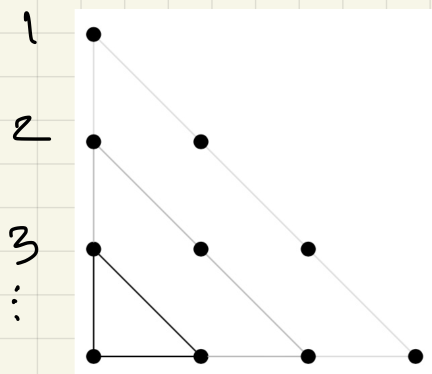
  $D(P)$ is a period of L_P

- We say that period collapse when the min. period is strictly less than $D(P)$.

Ex (standard simplex)

$\Gamma = \mathbb{Z}^2$, $P = \{(x_1, x_2) \in \mathbb{R}^2 \mid \begin{array}{l} x_1, x_2 \geq 0 \\ x_1 + x_2 \leq 1 \end{array} \} \subseteq \mathbb{R}^2$

$qP = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0,\ x_1 + x_2 \leq q\}$



$$P = \mathrm{conv}\{(0,0), (1,0), (0,1)\} \subseteq \mathbb{R}^2$$

$$L_P(q) = \frac{(q+1)(q+2)}{2}$$

More generally, $P = \{(x_1, \dots, x_\ell) \in \mathbb{R}^\ell \mid x_1 + \dots + x_\ell \leq 1 \text{ and all } x_i \geq 0\}$

then $L_P(q) = \binom{q+\ell}{\ell}$

Ex $\quad \Gamma = \mathbb{Z}^2 \quad P = [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}] \subseteq \mathbb{R}^2$

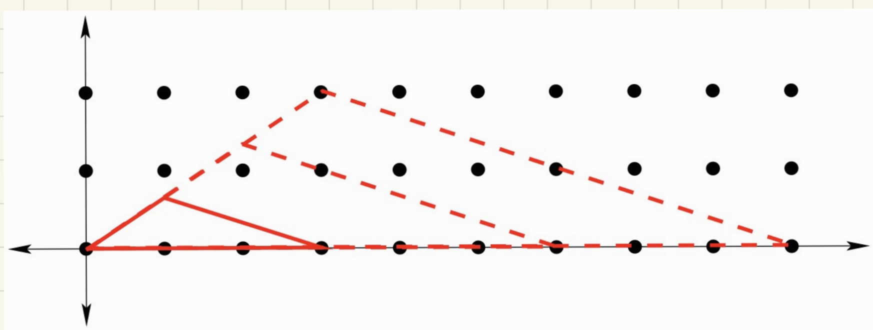
$$L_P(q) = \begin{cases} q^2 & q \equiv 1 \bmod 2 \\ (q+1)^2 & q \equiv 2 \bmod 2 \end{cases}$$

Ex (McAllister - Wood 2005)

$$P = \mathrm{conv}\{(0,0), (D,0), (1, \tfrac{D-1}{D})\} \subseteq \mathbb{R}^2 \quad D \geq 2$$

Then $\quad D(P) = D$ while

$$L_P(q) = \frac{D-1}{2} q^2 + \frac{D+1}{2} q + 1.$$



$$P = \mathrm{conv}\{(0,0), (3,0), (1, \tfrac{2}{3})\}$$

$D(P) = 3$ and $L_P(q) = (q+1)^2$

Def (hyperplane and subgroup arr.)

- Let $\Gamma = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i \simeq \mathbb{Z}^\ell$ free abelian group.
- Let $\mathcal{L}$ be a finite list (multiset) in Γ.
- $q \in \mathbb{Z}_{>0}$, $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$
- For $\alpha = \sum_{i=1}^{\ell} a_i \alpha_i \in \mathcal{L}$

  hyperplane $\quad H_{\alpha,\mathbb{R}} := \{ x \in \mathbb{R}^\ell \mid \sum_{i=1}^{\ell} a_i x_i = 0 \} \subseteq \mathbb{R}^\ell$

  subgr. $\quad H_{\alpha,\mathbb{Z}_q} := \{ z \in \mathbb{Z}_q^\ell \mid \sum_{i=1}^{\ell} a_i z_i = \bar{0} \} \subseteq \mathbb{Z}_q^\ell$
- The list $\mathcal{L}$ determines

  $\mathbb{R}$-arr. $\quad \mathcal{L}(\mathbb{R}) := \{ H_{\alpha,\mathbb{R}} \mid \alpha \in \mathcal{L} \}$

  $\mathbb{Z}_q$-arr. $\quad \mathcal{L}(\mathbb{Z}_q) := \{ H_{\alpha,\mathbb{Z}_q} \mid \alpha \in \mathcal{L} \}$

Def (Combinatorics of hyp. arr.)

- intersection poset $\quad L_{\mathcal{L}(\mathbb{R})} := \{ \bigcap_{\alpha \in S} H_{\alpha,\mathbb{R}} \mid S \subseteq \mathcal{L} \}$
- Möbius function: $\mu : L_{\mathcal{L}(\mathbb{R})} \longrightarrow \mathbb{Z}$

$$\mu(\mathbb{R}^\ell) := 1, \quad \mu(X) := - \sum_{X \subsetneq Y \subseteq \mathbb{R}^\ell} \mu(Y)$$

- Characteristic pol

$$\chi_{\mathcal{L}(\mathbb{R})}(t) := \sum_{X \in L_{\mathcal{L}(\mathbb{R})}} \mu(X) t^{\dim X}$$

Thm (Kamiya – Takemura – Terao 2008)

If $q \in \mathbb{Z}_{>0}$, then

$$\chi^{\text{quasi}}_{\mathcal{L}}(q) := \# \left(\mathbb{Z}_q^{\ell} \setminus \bigcup_{\alpha \in \mathcal{L}} H_{\alpha, \mathbb{Z}_q} \right)$$

is a monic quasi-pol in q of deg. ℓ with coefficients in $\mathbb{Z}$.

It's called the characteristic quasi-pol of $\mathcal{L}$ w.r.t. the group Γ.

Thm ("Finite field method")

The first constituent of $\chi^{\text{quasi}}_{\mathcal{L}}(q)$ coincides with the characteristic pol. of $\mathcal{L}(\mathbb{R})$

i.e. $\quad f^1_{\mathcal{L}}(t) = \chi_{\mathcal{L}(\mathbb{R})}(t)$

Ex $\quad \mathcal{L} = \{(-1,1),(0,2),(0,4)\} \subseteq \mathbb{Z}^2$

$$\mathcal{L}(\mathbb{R}) = \{\{-x_1+x_2=0\},\{x_2=0\},\{x_2=0\}\}$$

$$\begin{tikzcd}[column sep=small]
 & \{(0,0)\} \;\; 1 & \\
-1 \;\; \{-x_1+x_2=0\} \arrow[ur, dash] & & \{x_2=0\} \;\; -1 \arrow[ul, dash] \\
 & \mathbb{R}^2 \;\; 1 \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

$$\chi_{\mathcal{L}(\mathbb{R})}(t) = \boxed{t^2-2t+1}$$

$$\mathcal{L}(\mathbb{Z}_q) = \{\{-z_1+z_2=\bar{0}\},\{2z_2=\bar{0}\},\{4z_2=\bar{0}\}\}$$

$$\chi^{\text{quasi}}_{\mathcal{L}}(q) = \#\{z \in \mathbb{Z}_q^2 \mid -z_1+z_2,\ 2z_2,\ 4z_2 \not\equiv 0 \bmod q\}$$

$$= \begin{cases} \boxed{q^2-2q+1} & q \equiv 1,3 \bmod 4 \\ q^2-3q+2 & q \equiv 2 \bmod 4 \\ q^2-5q+4 & q \equiv 4 \bmod 4 \end{cases}$$

Rem (T-Yoshinaga, Liu-T-Yoshinaga)

Every constituent can be described as the "characteristic pol." of an arr.

Thm (Kamiya–Takemura–Terao 2008)

For a sublist $S \subseteq \mathcal{L}$, suppose

$$\text{tor. subgr.}\left(\mathbb{Z}^{\ell}/\langle S\rangle\right) \simeq \bigoplus_{i=1}^{n_S} \mathbb{Z}/d_{S,i}\mathbb{Z} \qquad n_S \geq 0$$

$$1 < d_{S,i} \mid d_{S,i+1}$$

Then $\rho_{\mathcal{L}} := \mathrm{lcm}\left(d_{S,n_S} \mid S \subseteq \mathcal{L}\right)$

is a period of $\chi^{\mathrm{quasi}}_{\mathcal{L}}$. (LCM-period)

Question Is $\rho_{\mathcal{L}}$ the minimum period of $\chi^{\mathrm{quasi}}_{\mathcal{L}}$?

(True, if $\mathcal{L} = \Phi^{+}$ positive system of a root system)

If not, study the "period collapse".

Two proofs of KTT:

1st:

$$\chi^{\mathrm{quasi}}_{\mathcal{L}}(q) = \sum_{S \subseteq \mathcal{L}} (-1)^{\#S} \left(\prod_{i=1}^{n_S} \gcd(d_{S,i}, q)\right) q^{\ell - \mathrm{rank}\langle S\rangle}$$

a quasi-pol. with min. period d_{S,n_S}

2nd via Ehrhart theory.

For $\alpha = \sum_{i=1}^{\ell} a_i \alpha_i \in \mathcal{L}$ write $c_\alpha := (a_1, \dots, a_\ell)^T \in \mathbb{Z}^\ell$

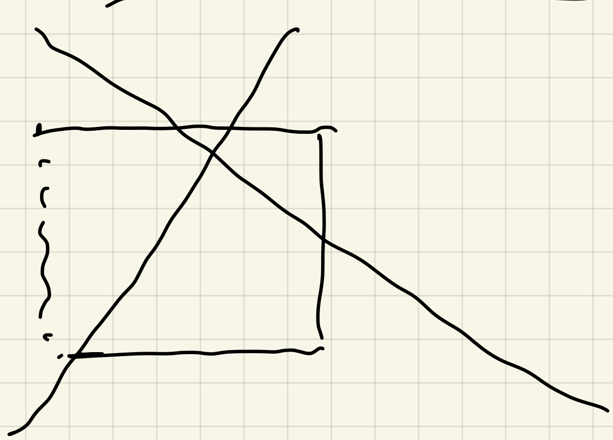
Use $(0,q] \cap \mathbb{Z} \simeq \mathbb{Z}_q$, we can write

$$\chi^{quasi}_{\mathcal{L}}(q) = \#\Big(\mathbb{Z}^\ell \cap (0,q]^\ell \setminus \bigcup_{\substack{\alpha\in\mathcal{L}\\ k\in\mathbb{Z}}} \{x \in \mathbb{R}^\ell \mid x\cdot c_\alpha = kq\}\Big)$$

$$= \#\Big(\mathbb{Z}^\ell \cap \Big(q \times \Big((0,1]^\ell \setminus \bigcup_{\substack{\alpha\in\mathcal{L}\\ k\in\mathbb{Z}}} \{x\in\mathbb{R}^\ell \mid x\cdot c_\alpha = k\}\Big)\Big)\Big)$$

$= \sum$ Ehrhart quasi-pol. of rational half-open polytopes

$= \sum$ Ehrhart quasi-pol. of a rational inside-out polytope Beck–Zaslavsky 2006



2. Free and supersolvable arr.

- let $\mathbb{K}$ be a field and $V = \mathbb{K}^\ell$
- A <u>hyperplane</u> in V is a 1-codim. subspace of V
- $\mathcal{A}$: a <u>central</u> arr. (a finite set of hyp.) in V.

<u>Def</u> (Free arr. Terao 1980)

- $\{x_1, \dots, x_\ell\}$ a basis $V^* = \mathrm{Hom}(V, \mathbb{K})$

  $S := \mathbb{K}[x_1, \dots, x_\ell]$

- For $H \in \mathcal{A}$, fix α_H s.t. $H = \ker \alpha_H$

  $$\alpha_H = a_1 x_1 + \dots + a_\ell x_\ell \neq 0 \quad a_i \in \mathbb{K}$$

- A <u>derivation</u> of S: a $\mathbb{K}$-linear map $\theta : S \to S$

  s.t. $\theta(fg) = f\theta(g) + g\theta(f) \quad \forall f, g \in S$

- The set of derivations $\mathrm{Der}(S)$ is a free S-module

  $$\mathrm{Der}(S) = \bigoplus_{i=1}^{\ell} S \frac{\partial}{\partial x_i}$$

  $$\left(\theta = \sum \theta(x_i) \frac{\partial}{\partial x_i} \right)$$

- The <u>module of $\mathcal{A}$-derivations</u>

$$D(\mathcal{A}) := \{ \theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in \alpha_H S \ \forall H \in \mathcal{A} \}$$
$$\subseteq \mathrm{Der}(S)$$

- $\mathcal{A}$ is a <u>free arr.</u> if $D(\mathcal{A})$ is a free S-module.

<u>Def</u> (Supersolvable arr., Stanley 1972, Björner - Edelman - Ziegler 1990)

- $\mathrm{rank}(\mathcal{A}) := \mathrm{codim} \bigcap_{H \in \mathcal{A}} H$
- $\mathcal{B} \subseteq \mathcal{A}$ is a <u>modular coatom</u> of $\mathcal{A}$ if

 (1) $\mathrm{rank}(\mathcal{B}) = \mathrm{rank}(\mathcal{A}) - 1$

 (2) for any $H \neq H' \in \mathcal{A} \setminus \mathcal{B}$, $\exists H'' \in \mathcal{B}$ s.t. $H \cap H' \subseteq H''$

- $\mathcal{A}$ is called <u>supsolvable</u> if $\exists$ chain of arr.

$$\emptyset = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \cdots \subseteq \mathcal{A}_r = \mathcal{A}$$

$\mathcal{A}_i$ is a modular coatom of $\mathcal{A}_{i+1}$ $\quad 0 \leq i \leq r-1$

Thm (Jambu-Terao 1984)

If $\mathcal{A}$ is supersolvable, then $\mathcal{A}$ is free

- $V = \mathbb{R}^\ell$ with standard inner product $(\cdot,\cdot)$
- $V \supseteq \Phi$: irreducible root system.
- $\Phi \supseteq \Phi^+$: positive system of Φ
- $\Phi^+ \supseteq \Delta = \{\alpha_1, \dots, \alpha_\ell\}$: base w.r.t. Φ^+
- $(\Phi^+, \geq)$: root poset $\beta_1, \beta_2 \in \Phi^+$

$$\beta_1 \geq \beta_2 \iff \beta_1 - \beta_1 \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0} \alpha_i$$

- $I \subseteq \Phi^+$ is called an ideal if $\beta_1, \beta_2 \in \Phi^+$

$$\beta_1 \geq \beta_2, \ \beta_1 \in I \implies \beta_2 \in I.$$

- $\mathcal{A}_{\Phi^+} := \{ H_\alpha \mid \alpha \in \Phi^+ \}$ Weyl arr.

$$H_\alpha = \{ x \in V \mid (\alpha, x) = 0 \}$$

$\mathcal{A}_\Psi = \{ H_\alpha \mid \alpha \in \Psi \}$ $\quad \Psi \subseteq \Phi^+$

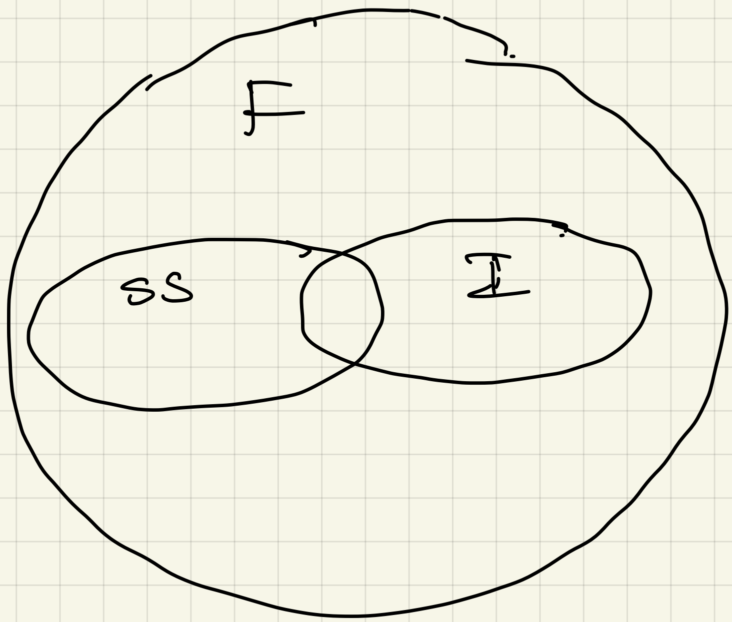
Weyl subarr.

$$\mathcal{A}_\Psi \underset{\text{affine equivalent}}{\sim} \Psi(\mathbb{R})$$

- $I \subseteq \Phi^+$, $\mathcal{A}_I$: ideal subarr.

<u>Thm</u> (Abe - Barakat - Cuntz - Hoge - Terao 2016)

If $I \subseteq \Phi^+$ is an ideal, then $\mathcal{A}_I$ is free.



F: free   SS: supersolvable   I: ideal subarr.

II, Geometry and Enumeration on Weyl arr.

1 Warpitzky-compatible arr.

- $\mathbb{R}^\ell = V \supseteq \Phi \supseteq \Phi^+ \supseteq \Delta$: as before
- $Q(\Phi) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$: root lattice
- $\Psi \subseteq \Phi^+ \subseteq Q(\Phi)$:

  $\chi^{\text{quasi}}_{\Psi}(q)$: the characteristic quasi-pol. of Ψ w.r.t. root lattice

Ex: $\Phi = A_2 = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$

$\Phi^+(A_2)$

$$(\alpha_1,\alpha_1) = (\alpha_2,\alpha_2) = -2(\alpha_1,\alpha_2)$$

$$\chi^{\text{quasi}}_{\emptyset}(q) = \#\mathbb{Z}_q^2 = q^2$$

$$\chi^{\text{quasi}}_{\{\alpha_1+\alpha_2\}}(q) = \#\{z \in \mathbb{Z}_q^2 \mid z_1+z_2 \neq \bar{0}\} = q(q-1)$$

$$\chi^{\text{quasi}}_{\Phi^+}(q) = \#\{z \in \mathbb{Z}_q^2 \mid z_1, z_1+z_2, z_2 \neq \bar{0}\} = (q-1)(q-2)$$

- $\Phi^+ \ni \tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i$: highest root w.r.t $(\Phi^+, \geq 0)$

  $\alpha_0 = -\tilde{\alpha}$, $c_0 := 1$

- $h = c_0 + c_1 + \cdots + c_\ell$: Coxeter number
- $f = \#\{0 \leq i \leq \ell \mid c_i = 1\}$: index of connection
- $W = \langle s_i \mid \alpha_i \in \Delta \rangle$: Weyl group.

  $s_i = s_{\alpha_i}$: simple reflection

- $k \in \mathbb{Z}$, $\alpha \in \Phi$, $H_{\alpha,k} := \{x \in V \mid (\alpha, x) = k\}$

  affine hyperplane.

- A connected component of $V \setminus \bigcup_{\substack{\alpha \in \Phi^+ \\ k \in \mathbb{Z}}} H_{\alpha,k}$

  is called an <u>alcove</u>

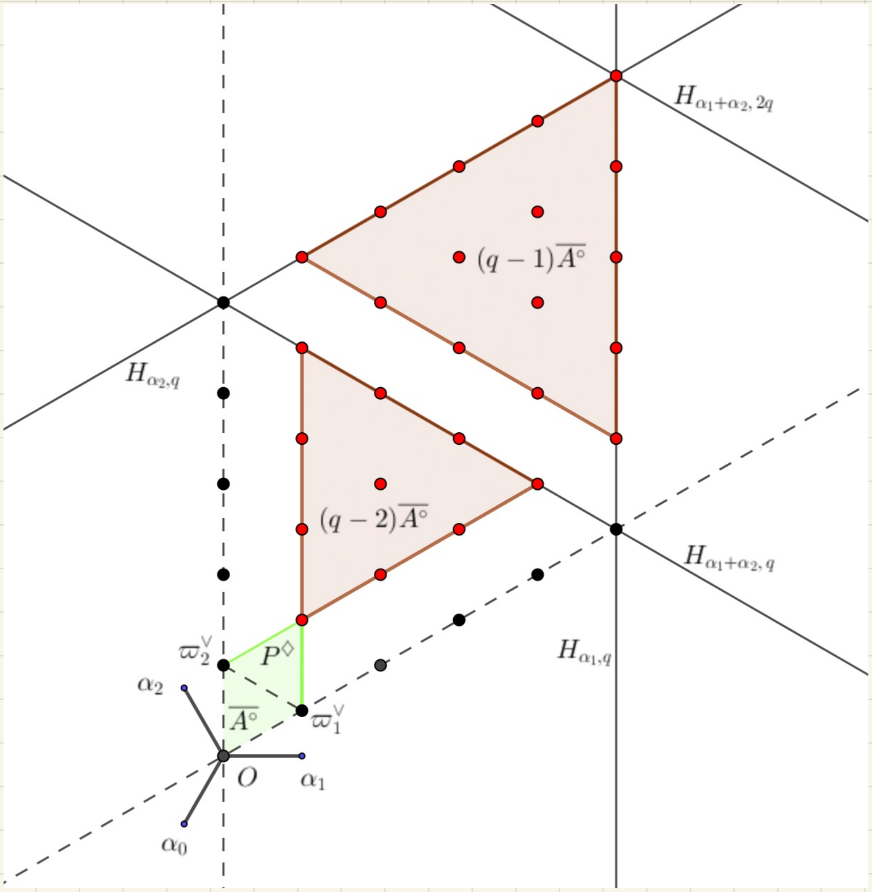
- $\{\varpi_1^\vee, \ldots, \varpi_\ell^\vee\}$: the <u>dual basis</u> of Δ

  $(\alpha_i, \varpi_j^\vee) = \delta_{ij}$

- $Z(\Phi^\vee) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\varpi_i^\vee$: coweight lattice
- $P^\diamond = \sum_{i=1}^{\ell} (0,1]\varpi_i^\vee$: fundamental parallelepiped
- $\overline{P^\diamond} \supseteq \overline{A^\circ} := \operatorname{conv}\left\{0, \frac{\varpi_1^\vee}{c_1}, \ldots, \frac{\varpi_\ell^\vee}{c_\ell}\right\}$: fundamental alcove.

- $L_{\overline{A^\circ}}(q) := \#\left(q\overline{A^\circ} \cap Z(\Phi^\vee)\right)$

Ehrhart quasi-pol. of $\overline{A^\circ}$ w.r.t. coweight lattice.



<u>Ex</u>: $\Phi = A_2$ $\quad W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^3 = e\rangle$

$\simeq \mathfrak{S}_3$

$Z(\Phi^\vee) = \mathbb{Z}\varpi_1^\vee \oplus \mathbb{Z}\varpi_2^\vee$

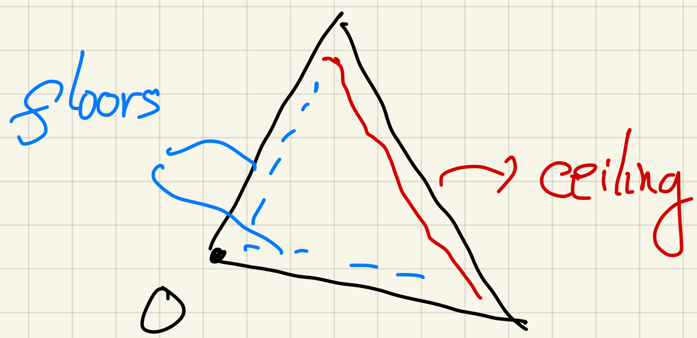
$\overline{A^\circ} = \mathrm{conv}\{0, \varpi_1^\vee, \varpi_2^\vee\}$: standard simplex in $\mathbb{R}\varpi_1^\vee \oplus \mathbb{R}\varpi_2^\vee$

$$L_{\overline{A^\circ}}(q) = \frac{(q+1)(q+2)}{2}$$

Def: A : alcove

The ceilings of A are the walls which do not pass through the origin and have the origin on the same side as A

supporting hyp. to a facet

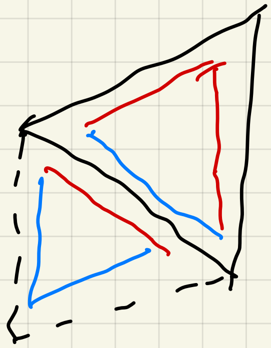


The upper closure

$$A^{\diamond} := A \cup (\text{ceiling facets of } A)$$

Thm (Worpitzky partition, Humphreys. Yoshinaga)

$$P^{\diamond} = \bigsqcup_{\substack{A:\text{ alcove} \\ A \subseteq P^{\diamond}}} A^{\diamond}$$



$P^\diamondsuit$

red: ceilings

blue: floors

(Ashraf – T – Yoshinaga)

Def: A $\Psi \subseteq \Phi^+$ is said to be <u>Worpitzky-compatible</u> (<u>or compatible</u>) if

$A^\diamondsuit \cap H_{\alpha, m_\alpha}$ is either empty or contained in a ceiling H_{β, m_β} of A with $\beta \in \Psi$, $m_\beta \in \mathbb{Z}$

$\alpha \in \Psi$

$m_\alpha \in \mathbb{Z}$

If Ψ is compatible, then we call $\mathcal{A}_\Psi$ compatible as well.

Def. (T-Tsuchiya)

A subset $\Psi \subseteq \Phi^+$ is said to be strongly (Worpitzky) compatible if for any $\alpha \in \Psi$ and for every choice of $\beta_1, \ldots, \beta_m \in \Phi^+$ s.t. $\alpha = \sum_{i=1}^{m} \mathbb{Z}_{>0} \beta_i$ then there exists k with $1 \le k \le m$ s.t. $\beta_k \in \Psi$.

Rem "there exists" $\rightsquigarrow$ "for all"

we obtain an equivalent def. of ideals

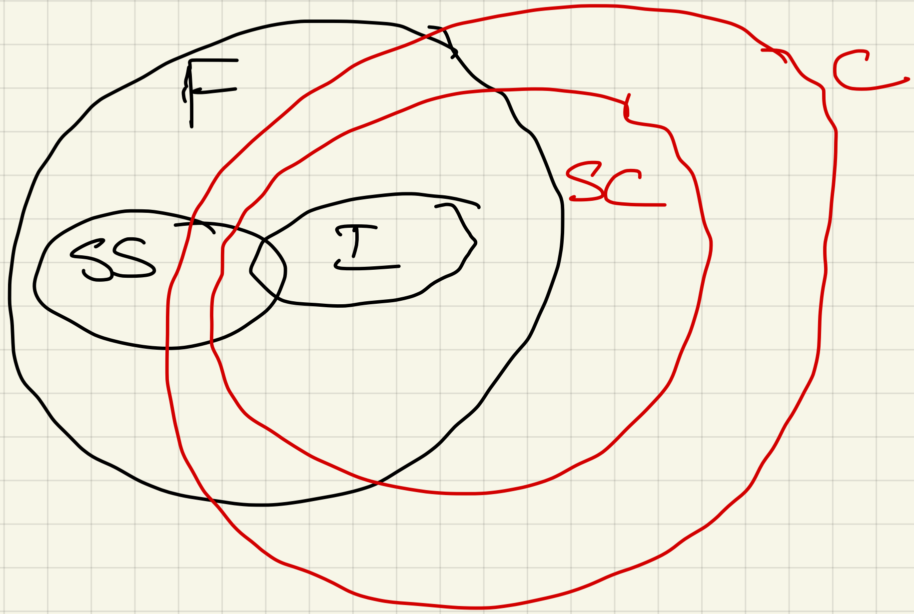
let $\mathcal{I}$ be the set of all ideals of Φ^+

C —— compatible sets of Φ^+

SC —— strongly compatible sets of Φ^+

Thm (Ashraf – T – Yoshinaga 2020)

$$\underset{\text{obvious}}{\mathcal{I} \subseteq} SC \subseteq C$$



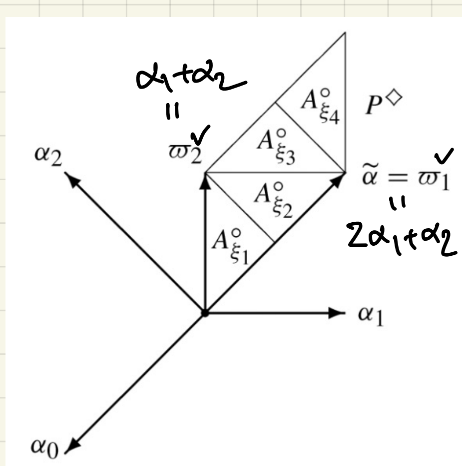
<u>Ex</u>: (a) $\emptyset, \Phi^+ \in C$.

If $\delta \in \Phi^+$, then $\Psi = \Phi^+ \setminus \{\delta\} \in C$

(b) $\Psi_1 = \{\alpha\}$ wher $\alpha = \sum_{i=1}^{\ell} d_i \alpha_i$ with all $d_i \geq 1$

then $\Psi_1 \notin C$

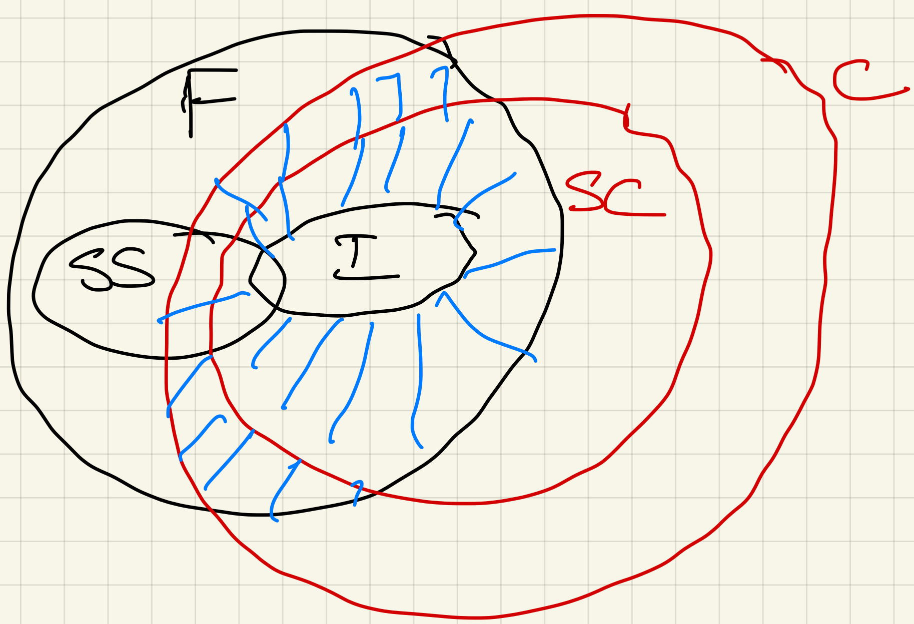
$\Phi = B_2$, $\Psi = \{\alpha_1+\alpha_2, \alpha_2+2\alpha_1\} \notin C$



Ex: $\Phi = B_2$ $\psi = \{\alpha_2, 2\alpha_1+\alpha_2\} \in C \setminus SC$

$\psi' = \{\alpha_1+\alpha_2, \alpha_1\} \in SC \setminus I$

Question: Is there any "significant" class of arr. contained in $F \cap C \setminus I$ or $F \cap SC \setminus I$



2. $\mathcal{A}$ - Eulerian polynomial

$$\tilde{\alpha} = \sum_{i=1}^{\ell} c_i \alpha_i \quad \text{highest root.}$$

$$\alpha_0 = -\tilde{\alpha}, \quad c_0 = 1$$

<u>Def:</u> (Ashraf - T - Yoshinaga)

let $\Psi \subseteq \Phi^+$, $\Psi^c := \Phi^+ \setminus \Psi$

- $w \in W$, $\operatorname{dsc}_\Psi(w) := \displaystyle\sum_{\substack{w(\alpha_i) \in -\Psi^c \\ 0 \le i \le \ell}} c_i$

- <u>$\mathcal{A}$ - Eulerian pol</u> of Ψ is defined by

$$E_\Psi(t) := \frac{1}{f} \sum_{w \in W} t^{h - \operatorname{dsc}_\Psi(w)}$$

<u>Thm:</u> $E_\Psi(t)$ has positive integer coefficient with no constant term.

<u>Thm</u> (Euler 1736) The classical ℓ-th Eulerian pol. is the polynomial $A_\ell(t)$

$$\sum_{q \ge 1} q^\ell t^q = \frac{A_\ell(t)}{(1-t)^{\ell+1}}$$

Rem:

- $\Psi = \Phi^+$, $dsc_{\Phi^+}(w) = 0 \quad \forall w \in W$

$$E_{\Phi^+}(t) = \frac{\# W}{S} t^h$$

- $\Psi = \emptyset \qquad E_{\emptyset}(t) = R_{\Phi}(t)$

Lam-Postnikov Eulerian polynomial

$$R_{\Phi}(t) = A_\ell(t) \cdot \prod_{i=1}^{\ell} \frac{1-t^{c_i}}{1-t}$$

$\Phi = A_\ell$, then $R_{\Phi}(t) = A_\ell(t)$

Def: $f : \mathbb{Z} \to \mathbb{C}$

$$P(S) = \sum_{k=1}^{n} a_k S^k \in \mathbb{C}[S]$$

shift operator $\quad (P(S)f)(t) := \sum_{k=1}^{n} a_k f(t-k)$

Thm (ATY) TFAE:

(i) $\Psi \subseteq \Phi^+$ is compatible

(ii) $$\chi_\Psi^{quasi}(q) = \left(E_\Psi(S)\, L_{\overline{A^\circ}}\right)(q)$$

(iii) $$\sum_{q \geq 1} \chi_\Psi^{quasi}(q)\, t^q = \frac{E_\Psi(t)}{\prod_{i=0}^{\ell} (1-t^{c_i})}$$

Rem: This recovers two known results:

$\Psi = \Phi^+$ Athanasiadis, Blass–Sagan, Suter, KTT

$\Psi = \emptyset$ Lam–Posnikov, Yoshinaga

Ex: $\Phi = A_2$

$$E_\emptyset(t) = t^2 + t = A_2(t)$$

$$E_{\{\alpha_1+\alpha_2\}}(t) = t^3 + t$$

$$E_{\Phi^+}(t) = 2t^3$$

$$L_{\overline{A0}}(q) = \frac{(q+1)(q+2)}{2}$$

$$\chi^{quasi}_{\phi}(q) = q^2 = \frac{(q-1)q}{2} + \frac{q(q+1)}{2} = ((S^2+S)L_{\overline{A0}})(q)$$

$$\Rightarrow \phi \in C$$

$$\chi^{quasi}_{\{\alpha_1+\alpha_2\}}(q) = q(q-1)$$

$$((S^3+S)L_{\overline{A0}})(q) = \frac{(q-1)(q-2)}{2} + \frac{q(q+1)}{2}$$

$$= q(q-1)+1$$

$$\Rightarrow \{\alpha_1+\alpha_2\} \notin C$$

Similarly, $\Phi^{+} \in C$.

3. Consideration on type A

1 Compatible arr. and cocomparability graphs

- $\{\varepsilon_1, \dots, \varepsilon_\ell\}$ an orthonormal basis for V

$$U := \left\{ \sum_{i=1}^{\ell} r_i \varepsilon_i \in V \;\middle|\; \sum_{i=1}^{\ell} r_i = 0 \right\} \simeq \mathbb{R}^{\ell-1}$$

- Set $\alpha_{ij} = \varepsilon_i - \varepsilon_j$

$\Phi(A_{\ell-1}) = \{ \pm\alpha_{ij} \mid 1 \le i < j \le \ell \}$ a rootsystem of type $A_{\ell-1}$

$\subseteq U$

$\Phi^+(A_{\ell-1}) = \{ \alpha_{ij} \mid 1 \le i < j \le \ell \}$

$\Delta(A_{\ell-1}) = \{ \alpha_{i\,i+1} \mid 1 \le i \le \ell-1 \}$

- $\Phi = A_{\ell-1}$; then $U \supseteq \mathcal{A}_{\Phi^+(A_{\ell-1})} = \underline{\mathrm{Br}(\ell)}$

Braid arr.

- $[\ell] = \{1, \dots, \ell\}$.

$$\{\text{subsets of } \Phi^+(A_{\ell-1})\} \xleftrightarrow{1-1} \{\text{subarr of } Br(\ell)\}$$

$$\updownarrow 1-1$$

$$\{\text{simple graphs on } [\ell]\}$$

$$\alpha_{ij} \in \Psi(G) \Longleftrightarrow H_{\alpha_{ij}} \in \mathcal{B} \Longleftrightarrow \{i,j\} \in E$$

$$G = ([\ell], E)$$

- For $G = ([\ell], E)$, let $\mathcal{A}(G) := \mathcal{A}_{\Psi(G)}$

$$\Psi(G) := \{ \alpha_{ij} \mid \{i,j\} \in E \ (i<j) \}$$

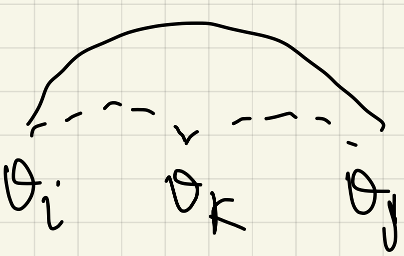
$\mathcal{A}(G)$: graphic arr. $\subseteq \mathbb{R}^{\ell-1}$

Def: A graph is called a chordal graph if it's C_{n+4} - free

Def: A graph is called a cocomparability graph if its complement has a transitive orientation

if $u \to v$ and $v \to w$ then $u \xrightarrow{\exists} w$

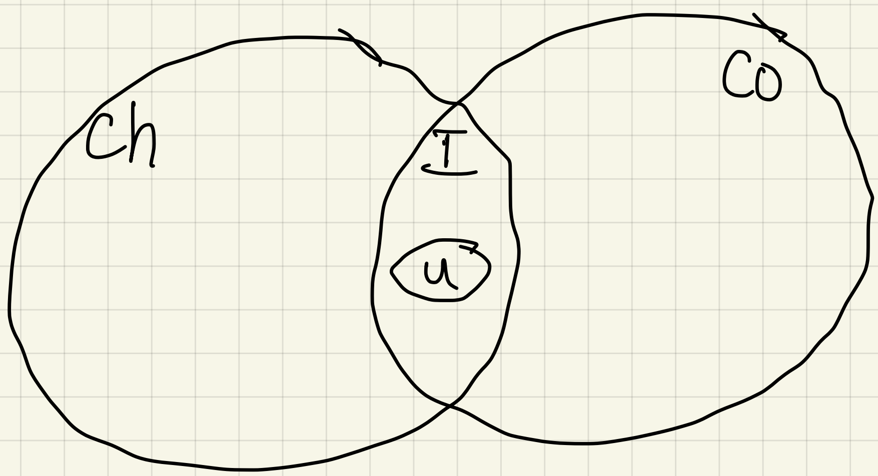
Equivalently $G = (V, E)$ with $|V| = \ell$ is cocomp. graph if it has an umbrella-free ordering : an ordering $v_1 < \dots < v_\ell$ of its vertices s.t. $i < k < j$, $(v_i, v_j) \in E$ then either $\{v_i, v_k\} \in E$ or $\{v_k, v_j\} \in E$ or both.



Def: A graph is called $\left|\begin{array}{l}\text{an interval graph} \\ \text{a unit interval graph}\end{array}\right|$ if

each vertex can be assoc. with $\left|\begin{array}{l}\text{an interval} \\ \text{a unit interval}\end{array}\right|$

on the real line, and two vertices are adjacent if the assoc. intervals have a nonempty intersection.



$$U \subsetneq I = Ch \cap Co.$$

Co: Cocomparability Ch: Chordal

I: interval U: unit interval.

Thm: (Stanley 1972, Edelman–Reiner 1994)

Let $G=([\ell],E)$ be a graph. TFAE:

(i) $\mathcal{A}(G)$ is free

(ii) $\mathcal{A}(G)$ is supersolvable

(iii) G is a chordal graph.

Thm (Folklore) $G=(V,E)$ $|V|=\ell$.

G has a labeling using $[\ell]$ so that $\mathcal{A}(G)$ is an ideal-graphic arr i.e. $\Psi(G)\subseteq\Phi^+(A_{\ell-1})$ is an ideal $\Longleftrightarrow$ G is a unit interval graph.

<u>Thm</u>. (T–Tsuchiya) $G=(V,E)$ $|V|=\ell$

G has a labeling from $[\ell]$ so that .

$$\mathcal{A}(G) \text{ is } \left|\begin{array}{l}\text{compatible}\\ \text{compatible and free}\end{array}\right.$$

$$\Longleftrightarrow \quad G \text{ is } \begin{array}{r}\text{a}\\ \text{an}\end{array}\left|\begin{array}{l}\text{cocomparability graph.}\\ \text{interval graph.}\end{array}\right.$$

Graph class	Weyl subarr. class	Ref.
Cocomp.	compatible (= strongly compatible)	T–T
Chordal	free (= supersolvable)	Stanley, E–R.
interval	compatible ∩ free	T–T
unit interval	ideal	Folklore

Parallel concepts in type A

<u>Thm</u> If $\Phi = A_\ell$, then $SC = C$.

2. Application to graph pol.

G: simple graph.

$C_G(t)$: chrometic pol.

The graphic Eulerian pol $W_G(t)$:

$$\sum_{q \geq 0} C_G(q)\, t^q = \frac{W_G(t)}{(1-t)^{\ell+1}}$$

The reduced graphic Eulerian pol $Y_G(t)$

$$\sum_{q \geq 1} \frac{C_G(q)}{q}\, t^q = \frac{Y_G(t)}{(1-t)^{\ell}}$$

Fact: $W_G(t)$, $Y_G(t)$ both have positive integer coefficient

Def: $G = ([\ell], E)$: simple graph

$\pi = \pi_1 \cdots \pi_\ell \in \mathfrak{S}_\ell$ has a $\mathcal{A}$-descent (w.r.t. G) at $i \in [\ell]$ if

- $\pi_i > \pi_{i+1}$ and
- $\{\pi_i, \pi_{i+1}\} \in E(G^c)$ $\quad (\pi_{\ell+1} = \pi_1)$

complement

Define $F_G(t) := \sum_{k=1}^{\ell} f_k(G)\, t^k$

$$f_k(G) := \frac{1}{\ell} \# \{ \pi \in \mathfrak{S}_\ell \mid \pi \text{ has } \ell - k \ \mathcal{A}\text{-descents} \}$$

Prop: $F_G(t) = \mathcal{A}$-Eulerian pol. of $\Psi(G)$

$$\Psi(G) = \{ \varepsilon_i - \varepsilon_j \mid \{i, j\} \in E,\ i < j \}$$

Thm: $G = ([\ell], E)$. TFAE.

(i) G is a cocomparability graph and

$1 < 2 < \cdots < \ell$ is umbrella-free

(ii) $1 < 2 < \cdots < \ell$ is umbrella-free

(iii) $\Psi(G) \in \Phi^{\dagger}(A_{\ell-1})$ is compatible (= strongly compatible)

(iv) $$C_G(t) = t \sum_{k=1}^{n} f_k(G) \binom{t+\ell-1-k}{\ell-1}$$

(v) $$F_G(t) = Y_G(t)$$

(vi) $$W_G(t) = t(1-t)\frac{d}{dt} F_G(t) + t\ell F_G(t)$$

$\Leftrightarrow \forall\, 1 \le k \le \ell,\ w_k(G) = k f_k(G) + (\ell-k+1) f_{k-1}(G)$

<u>Question</u> (Brenti 1992)

Is $W_G(t)$ log-concave (or just unimodal) ?

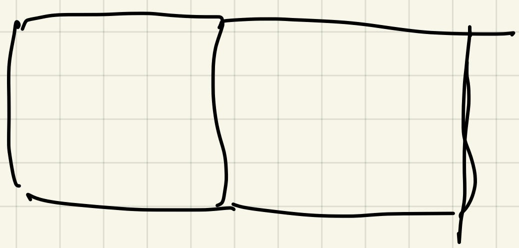
<u>Prop</u>: If $Y_G(t)$ is log-concave $\Rightarrow$ $W_G(t)$ is also log-concave

Conj: If G is a cocomparability graph, then $Y_G(t)$ is log-concave

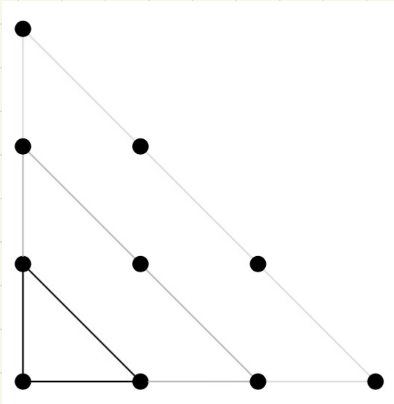
$\Longleftrightarrow$ If G is a cocomp. graph and $1 < 2 < \cdots < l$ is umbrella-free

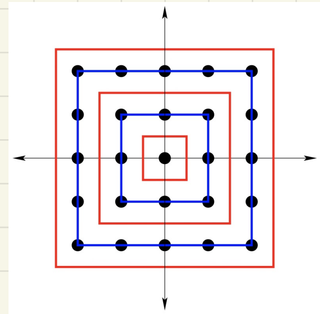
then $F_G(t)$ is log-concave.

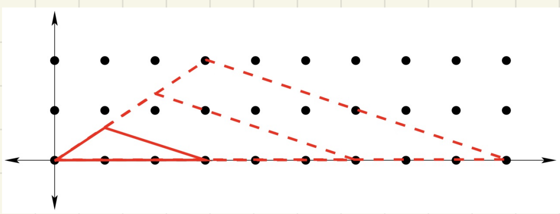
Rem: (J-T) Conj. holds true for cocomp. graphs on ≤ 8 vertices

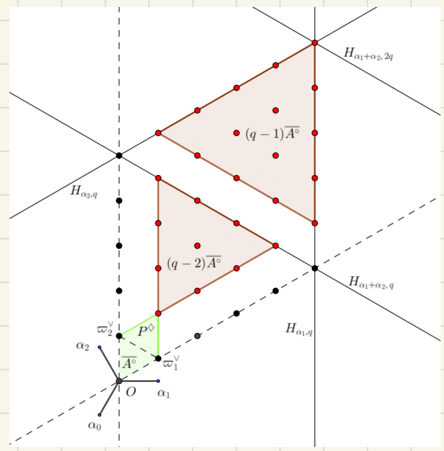


W_G Y_G is not real-rooted

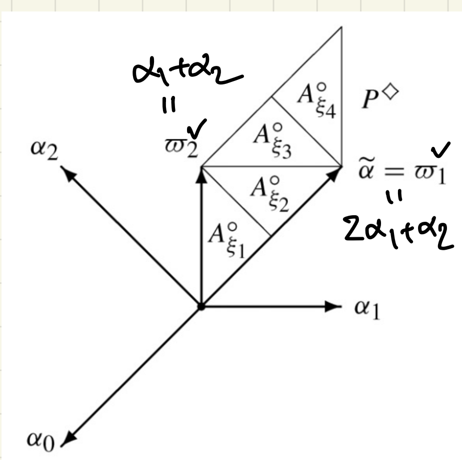




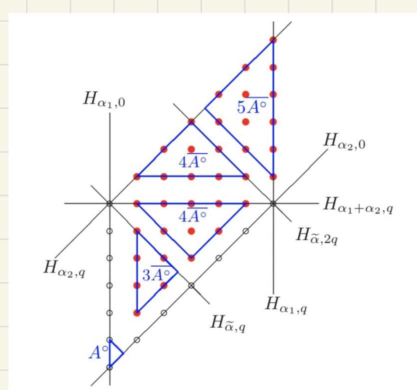




$H_{\alpha_1+\alpha_2,2q}$
$(q-1)\overline{A^\circ}$
$H_{\alpha_2,q}$
$(q-2)\overline{A^\circ}$
$H_{\alpha_1+\alpha_2,q}$
$H_{\alpha_1,q}$
$P^\diamond$
$\varpi_2^\vee$
α_2
$\overline{A^\circ}$
$\varpi_1^\vee$
O
α_1
α_0



$\alpha_1+\alpha_2$
$\varpi_2^\vee$
$A^\circ_{\xi_4}$
$P^\diamond$
$A^\circ_{\xi_3}$
α_2
$A^\circ_{\xi_2}$
$\widetilde{\alpha} = \varpi_1^\vee$
$A^\circ_{\xi_1}$
$2\alpha_1+\alpha_2$
α_1
α_0



$H_{\alpha_1,0}$
$5\overline{A^\circ}$
$H_{\alpha_2,0}$
$4\overline{A^\circ}$
$4\overline{A^\circ}$
$H_{\alpha_1+\alpha_2,q}$
$H_{\widetilde{\alpha},2q}$
$H_{\alpha_2,q}$
$3\overline{A^\circ}$
$H_{\alpha_1,q}$
$H_{\widetilde{\alpha},q}$
A°