# Locally anti-blocking lattice polytopes and their $h^*$-polynomials

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- OT1 Enriched chain polytopes, *Israel J. Math.*, to appear.
- OT2 The $h^*$-polynomials of locally anti-blocking lattice polytopes and their $\gamma$-positivity, arXiv:1906.04719.
- OT3 Reflexive polytopes arising from bipartite graphs with $\gamma$-positivity associated to interior polynoials, arXiv:1810.12258.

1. Unimodality, $\gamma$-positivity and real-rootedness
2. Anti-blocking polytopes, unconditional polytopes and locally anti-blocking polytopes
3. Chain polytopes, enriched chain polytopes and twinned chain polytopes
4. Symmetric edge polytopes of types A and B

## Palindromic polynomials and $\gamma$-positivity

$f(t) = \sum_{i=0}^{d} a_i t^i \in \mathbb{Z}_{>0}[t]$ : a palindromic polynomial

i.e., $a_i = a_{d-i}$ for any $1 \leq i \leq \lfloor d/2 \rfloor$

Then there exists a unique expression

$$f(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1+t)^{d-2i}$$

$\gamma(t) := \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i \in \mathbb{Z}[t]$ is called the $\gamma$-polynomial of $f(t)$.

(RR) $f(t)$ is real-rooted if all roots of $f(t)$ are real.

(GP) $f(t)$ is $\gamma$-positive if $\gamma_i \geq 0$ for all $i$.

(UN) $f(t)$ is unimodal if $a_0 \leq \cdots \leq a_k \geq \cdots \geq a_d$ with some $k$.

In general, (RR) $\Rightarrow$ (GP) $\Rightarrow$ (UN). If $f(t)$ is $\gamma$-positive, then

$$f(t) \text{ is real-rooted} \iff \gamma(t) \text{ is real-rooted}$$

## Gal Conjecture

**Conjecture (Real Root Conjecture, disproved)**

*The $h$-polynomial of a flag triangulation of a sphere is real-rooted.*

Gal found a counterexample for the Real Root Conjecture.

**Conjecture (Gal Conjecture)**

*The $h$-polynomial of a flag triangulation of a sphere is $\gamma$-positive.*

**Conjecture (Nevo-Petersen Conjecture)**

*The $\gamma$-polynomial of the $h$-polynomial of a flag triangulation of a sphere coincides with the $f$-polynomial of a flag simplicial complex.*

## Ehrhart theory

$\mathcal{P} \subset \mathbb{R}^d$ : a lattice polytope of dimension $d$
(i.e., a convex polytope all of whose vertices are in $\mathbb{Z}^d$)
$m\mathcal{P} = \{m\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$ : the $m$th dilated polytope of $\mathcal{P}$
$L_{\mathcal{P}}(m) := |m\mathcal{P} \cap \mathbb{Z}^d|$ : the Ehrhart polynomial of $\mathcal{P}$

## Theorem (Ehrhart)

*$L_{\mathcal{P}}(m)$ is a polynomial in $m$ of degree $d$.*

$\mathrm{Ehr}(\mathcal{P}, t) := 1 + \sum_{k=1}^{\infty} L_{\mathcal{P}}(m)t^m$ : the Ehrhart series of $\mathcal{P}$.

$(1-t)^{d+1}\mathrm{Ehr}(\mathcal{P}, t) = \sum_{i=0}^{d} h_i^* t^i =: h^*(\mathcal{P}, t)$ : the $h^*$-polynomial of $\mathcal{P}$.

## Remark

- $h_0^* = 1, h_1^* = |\mathcal{P} \cap \mathbb{Z}^d| - (d+1)$ *and* $h_d^* = |\mathrm{int}(\mathcal{P}) \cap \mathbb{Z}^d|$.
- *each* $h_i^* \geq 0$ *(Stanley).*
- $h^*(\mathcal{P}, 1)$ *equals the normalized volume of* $\mathcal{P}$.

## Reflexive polytope

$\mathcal{P} \subset \mathbb{R}^d$ : a lattice polytope of dimension $d$
$\mathrm{int}(\mathcal{P}) :=$ the interior of $\mathcal{P}$
$\mathbf{0} \in \mathbb{R}^d$ : the origin of $\mathbb{R}^d$

Assume that $\mathbf{0} \in \mathrm{int}(\mathcal{P})$.
$\mathcal{P}^\vee := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \leq 1$ for $\forall \mathbf{x} \in \mathcal{P}\}$ : the dual polytope of $\mathcal{P}$

**Definition**
We say that $\mathcal{P}$ is reflexive if $\mathbf{0} \in$ int$(\mathcal{P})$ and $\mathcal{P}^\vee$ is a lattice polytope.

**Definition**
We say that $\mathcal{P}$ is Gorenstein if $r\mathcal{P} + \mathbf{w}$ is reflexive for some $r \in \mathbb{Z}_{\geq 1}$ and $\mathbf{w} \in \mathbb{Z}^d$.

## Palindromic $h^*$-polynomials

**Theorem (Hibi)**

*A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ with $\mathbf{0} \in \mathrm{int}(\mathcal{P})$ is reflexive if and only if $h^*(\mathcal{P}, t)$ is palindromic and of degree $d$.*

**Theorem (De Negri-Hibi)**

*A lattice polytope $\mathcal{P}$ is Gorenstein if and only if $h^*(\mathcal{P}, t)$ is palindromic.*

**Theorem (Bruns-Römer)**

*If $\mathcal{P}$ is a Gorenstein polytope with a regular unimodular triangulation ($\iff$ the toric ideal of $\mathcal{P}$ has a squarefree initial ideal), then $h^*(\mathcal{P}, t)$ is unimodal.*

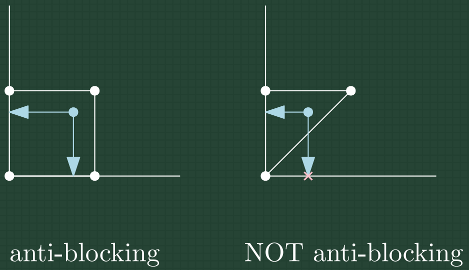
**Conjecture (Gal Conjecutre in Ehrhart theory)**

*The $h^*$-polynomial of a reflexive polytope with a central, flag, regular unimodular triangulation is $\gamma$-positive.*

## Anti-blocking polytopes

### Definition

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension $d$ is called anti-blocking if for any $\mathbf{y} = (y_1, \ldots, y_d) \in \mathcal{P}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d_{\geq 0}$ with $0 \leq x_i \leq y_i$ for all $i$, it holds that $\mathbf{x} \in \mathcal{P}$.



## Unconditional polytopes

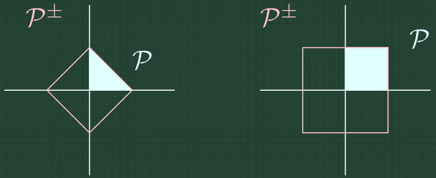
For $\varepsilon \in \{-1,1\}^d$ and $\mathbf{x} \in \mathbb{R}^d$, set $\varepsilon\mathbf{x} := (\varepsilon_1 x_1, \ldots, \varepsilon_d x_d) \in \mathbb{R}^d$.

### Definition (Kohl-Olsen-Raman)

Given an anti-blocking lattice polytope $\mathcal{P} \subset \mathbb{R}^d_{\geq 0}$ of dimension $d$, define

$$\mathcal{P}^{\pm} := \{\varepsilon\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \in \mathcal{P}, \varepsilon \in \{-1,1\}^d\}.$$

The polytope $\mathcal{P}^{\pm}$ is called an unconditional lattice polytope.

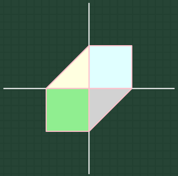


## Locally anti-blocking polytopes

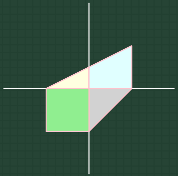
Given $\varepsilon \in \{-1,1\}^d$, define $\mathbb{R}^d_\varepsilon := \{\mathbf{x} \in \mathbb{R}^d : \varepsilon_i x_i \geq 0, ^\forall i\}$.

### Definition

A lattice polytope $\mathcal{P} \subset \mathbb{R}^d$ of dimension $d$ is called locally anti-blocking if for each $\varepsilon \in \{-1,1\}^d$, there exists an anti-blocking lattice polytope $\mathcal{P}_\varepsilon \subset \mathbb{R}^d_{\geq 0}$ of dimension $d$ such that $\mathcal{P} \cap \mathbb{R}^d_\varepsilon = \mathcal{P}^\pm_\varepsilon \cap \mathbb{R}^d_\varepsilon$. We call $\mathcal{P}_\varepsilon$ the anti-blocking piece on $\varepsilon$ of $\mathcal{P}$.



locally anti-blocking



NOT locally anti-blocking

### Remark

*Unconditional lattice polytopes are locally anti-blocking.*

# $h^*$-polynomials of locally anti-blocking lattice polytopes

**Theorem (Ohsugi-T)**

*Let $\mathcal{P} \subset \mathbb{R}^d$ be a locally anti-blocking lattice polytope of dimension $d$ and for each $\varepsilon \in \{-1,1\}^d$, let $\mathcal{P}_\varepsilon$ be the anti-blocking piece on $\varepsilon$. Then one has*

$$h^*(\mathcal{P}, t) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} h^*(\mathcal{P}_\varepsilon^{\pm}, t).$$

*In particular, $h^*(\mathcal{P}, t)$ is $\gamma$-positive if $h^*(\mathcal{P}_\varepsilon^{\pm}, t)$ is $\gamma$-positive for all $\varepsilon \in \{-1,1\}^d$.*

## Chain polytopes

$(P, <_P)$ : a poset on $[d] := \{1, \ldots, d\}$.

**Definition (Stanley)**

The chain polytope of $P$ is

$$\mathcal{C}_P := \{\mathbf{x} \in [0,1]^d : x_{i_1} + \cdots + x_{i_r} \leq 1 \text{ if } i_1 <_P \cdots <_P i_r \text{ is a chain in } P\}.$$

**Proposition**

*$\mathcal{C}_P$ is an anti-blocking lattice polytope with a flag regular unimodular triangulation.*

**Theorem (Stanley, Hibi)**

*$\mathcal{C}_P$ is Gorenstein if and only if $P$ is pure.*

## $P$-Eulerian polynomials

$(P, <_P)$ : a naturally labeled poset on $[d]$, i.e., $i <_P j \Rightarrow i < j$.
$\mathcal{L}(P)$: the set of linear extensions of $P$.
For $\pi \in \mathcal{L}(P)$, set

$$\mathrm{des}(\pi) := |\{1 \le i \le d-1 : \pi_i < \pi_{i+1}\}|.$$

$W_P(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{des}(\pi)}$ : the $P$-Eulerian polynomial.

**Theorem (Stanley)**

$$h^*(\mathcal{O}_P, t) = h^*(\mathcal{C}_P, t) = W_P(t).$$

**Theorem (Brändén)**

*If $P$ is pure, then $W_P(t)$ is $\gamma$-positive.*

**Conjecture (Stanley)**

*$W_P(t)$ is unimodal.*

## Enriched Chain polytopes

$P$ : a poset on $[d]$

**Definition (Ohsugi-T)**

The enriched chain polytope of $P$ is

$$\mathcal{C}_P^{(e)} := \mathcal{C}_P^{\pm}.$$

**Theorem (Ohsugi-T)**

*$\mathcal{C}_P^{(e)}$ is an unconditional reflexive polytope with a central flag regular unimodular triangulation.*

**Remark**

*We call $\mathcal{C}_P^{(e)}$ an enriched chain polytope because this polytope is related to the theory of enriched $P$-partitions introduced by John Stembridge.*

## Enriched Order Polytopes

$P$ : a naturally labeled poset on $[d]$.

$f : P \to \mathbb{Z}_{\geq 0}$ is called a $P$-partition if for any $i <_P j$,

- $f(i) \leq f(j)$.

$f : P \to \mathbb{Z}$ is called a left enriched $P$-partition if for any $i <_P j$,

- $|f(i)| \leq |f(j)|$;
- $|f(i)| = |f(j)| \Rightarrow f(j) \geq 0$.

The order polytope $\mathcal{O}_P$ of $P$ is the convex hull of $\{(f(1), \ldots, f(d)) : f \text{ is a } P\text{-partition with } f(i) \leq 1\}$.

The enriched order polytope $\mathcal{O}_P^{(e)}$ of $P$ is the convex hull of $\{(f(1), \ldots, f(d)) : f \text{ is a left enriched } P\text{-partition with } |f(i)| \leq 1\}$.

**Theorem (Stanley, Ohsugi-T)**

- $L_{\mathcal{O}_P}(t) = L_{\mathcal{C}_P}(t) = |\{P\text{-partitions } f \text{ with } f(i) \leq t\}|$
- $L_{\mathcal{O}_P^{(e)}}(t) = L_{\mathcal{C}_P^{(e)}}(t) = |\{\text{left enriched } P\text{-partitions } f \text{ with } |f(i)| \leq t\}|$

## Left peak polynomials

$(P, <_P)$ : a naturally labeled poset on $[d]$.
For $\pi \in \mathcal{L}(P)$ with $\pi_0 = 0$, set

$$\mathrm{peak}^{(\ell)}(\pi) := |\{1 \le i \le d-1 : \pi_{i-1} < \pi_i > \pi_{i+1}\}|.$$

$W_P^{(\ell)}(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\mathrm{peak}^{(\ell)}(\pi)}$ : the left peak polynomial of $P$.

**Theorem (Ohsugi-T, Petersen, Stembridge)**

$$h^*(\mathcal{O}_P^{(e)}, t) = h^*(\mathcal{C}_P^{(e)}, t) = (t+1)^d W_P^{(\ell)} \left( \frac{4t}{(t+1)^2} \right).$$

*Namely, the $\gamma$-polynomial equals $W_P^{(\ell)}(4t)$.*

**Theorem (Nevo-Petersen, Ohsugi-T)**

$W_P^{(\ell)}(4t)$ *coincides with the $f$-polynomial of a flag simplicial complex.*

## Twinned chain polytopes

For a lattice polytope $\mathcal{P} \subset \mathbb{R}^d$, write $-\mathcal{P} := \{-\mathbf{x} : \mathbf{x} \in \mathcal{P}\}$.
$P, Q$ : posets on $[d]$.

**Definition (Ohsugi-Hibi, Hibi-Matsuda-T)**

The twinned chain polytope of $P$ and $Q$ is

$$\mathcal{C}_{P,Q} := \mathrm{conv}(\mathcal{C}_P \cup (-\mathcal{C}_Q)).$$

**Theorem (Hibi-Matsuda-T,T)**

*$\mathcal{C}_{P,Q}$ is a locally anti-blocking reflexive polytope with a central flag regular unimodular triangulation. In particular, each anti-blocking piace is a chain polytope.*

## Twinned chain polytopes

For $W \subset [d]$, write $\overline{W} = [d] \setminus W$.
$P_W$ : the induced subposet of $P$ on $W$.

**Theorem (Ohsugi-T)**

*For each $\varepsilon \in \{-1, 1\}^d$, let $I_\varepsilon = \{i \in [d] : \varepsilon_i = 1\}$ and $R_\varepsilon$ a naturally labeled poset which is obtained from $P_{I_\varepsilon} \oplus Q_{\overline{I_\varepsilon}}$ by reordering the label. Set*

$$W_{P,Q}(t) = \frac{1}{2^d} \sum_{\varepsilon \in \{-1,1\}^d} W_{R_\varepsilon}^{(\ell)}(t).$$

*Then one has*

$$h^*(\mathcal{C}_{P,Q}, t) = (1+t)^d W_{P,Q}\left(\frac{4t}{(1+t)^2}\right).$$

*Namely, the $\gamma$-polynomial equals $W_{P.Q}(4t)$.*

## Symmetric edge polytopes of type B

$G$ : a simple graph on $[d]$ with edge set $E(G)$

**Definition (Ohsugi-T)**

The symmetric edge polytope of type B of $G$ is

$$\mathcal{B}_G := \mathrm{conv}(\{\pm\mathbf{e}_1, \ldots, \pm\mathbf{e}_d\} \cup \{\pm\mathbf{e}_i \pm \mathbf{e}_j : \{i,j\} \in E(G)\}).$$

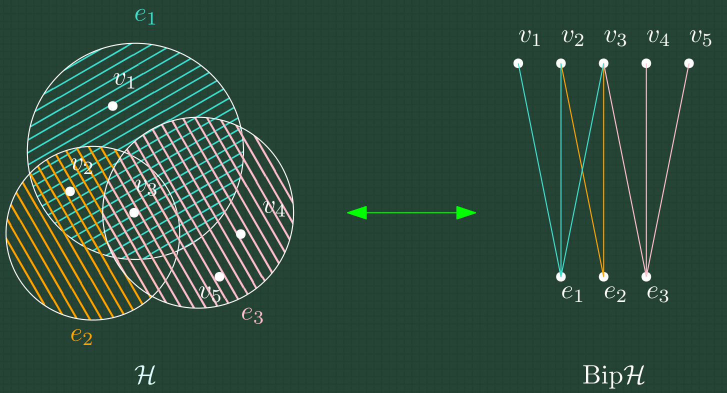
**Theorem (Ohsugi-T)**

*$\mathcal{B}_G$ is unconditional. Moreover, $\mathcal{B}_G$ is reflexive (with a regular unimodular triangulation) if and only if $G$ is bipartite. Furthermore, $\mathcal{B}_G$ is reflexive with a (central) flag regular unimodular triangulation if and only if $G$ is a chordal bipartite graph.*

## Interior polynomials of hypergraphs

A hypergraph is a pair $\mathcal{H} = (V, E)$, where $V = \{v_1, \ldots, v_m\}$ and $E = \{e_1, \ldots, e_n\} \subset 2^V \setminus \{\emptyset\}$.

Bip$\mathcal{H}$ is the bipartite graph with a bipartition $V \cup E$ such that $\{v_i, e_j\}$ is an edge of $\mathrm{Bip}\mathcal{H}$ if $v_i \in e_j$



## Interior polynomials of hypergraphs

Let $G = \mathrm{Bip}\mathcal{H}$ and assume that $G$ is connected.

A hypertree in $\mathcal{H}$ is a function $\mathbf{f} : E \to \mathbb{Z}_{\geq 0}$ such that there exists a spanning tree $\Gamma$ of $G$ such that $\deg(e) = \mathbf{f}(e) + 1$ for any $e \in E$.

$$B_{\mathcal{H}} := \{\text{hypertrees in } \mathcal{H}\}$$

$e_j \in E$ is said to be internally inactive with respect to $\mathbf{f} \in B_{\mathcal{H}}$ if there exist $\mathbf{f}' \in B_{\mathcal{H}}$ and $j' < j$ such that

$$\mathbf{f}'(e_i) = \begin{cases} \mathbf{f}(e_i) - 1 & (i = j) \\ \mathbf{f}(e_i) + 1 & (i = j') \\ \mathbf{f}(e_i) & (\text{otherwise}) \end{cases}$$

## Interior polynomials of hypergraphs

$e_j \in E$ is said to be internally inactive with respect to $\mathbf{f} \in B_{\mathcal{H}}$ if there exist $\mathbf{f}' \in B_{\mathcal{H}}$ and $j' < j$ such that

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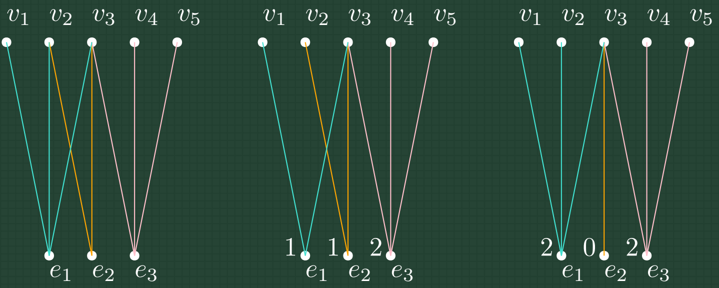



Bip$\mathcal{H}$ $e_2$ is internally inactive

## Interior polynomials of hypergraphs

$\bar{\iota}(\mathbf{f})$ : the number of internally inactive hyperedges of $\mathbf{f} \in B_{\mathcal{H}}$.
The interior polynomial of $\mathcal{H}$ is the generating function

$$I_{\mathcal{H}}(t) = \sum_{\mathbf{f} \in B_{\mathcal{H}}} t^{\bar{\iota}(\mathbf{f})}.$$

If $G$ is a connected bipartite graph such that $G = \mathrm{Bip}\mathcal{H}$ for a hyper graph $\mathcal{H}$, then we write $I_G(t) := I_{\mathcal{H}}(t)$.
For a connected bipartite graph $G$ on $[d]$, the edge polytope (or root polytope) of $G$ is

$$\mathcal{P}_G = \mathrm{conv}(\{\mathbf{e}_i + \mathbf{e}_j : \{i,j\} \in E(G)\}).$$

**Theorem (Kálmán-Postnikov)**

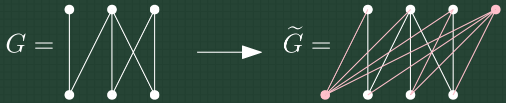
*For a connected bipartite graph $G$, one has*

$$h^*(\mathcal{P}_G, t) = I_G(t).$$

## The $\gamma$-positivity of $h^*(\mathcal{B}_G, t)$

$G$ : a bipartite graph with a bipartition $V_1 \cup V_2 = [d]$
Let $\widetilde{G}$ be the connected bipartite graph on $[d+2]$ whose edge set is

$$E(\widetilde{G}) = E(G) \cup \{\{d+1, d+2\}\} \cup \{\{i, d+1\} : i \in V_1\} \cup \{\{j, d+2\} : j \in V_2\}$$




### Theorem (Ohsugi-T)

$$h^*(\mathcal{B}_G, t) = (1+t)^d I_{\widetilde{G}}\left(\frac{4t}{(1+t)^2}\right)$$

*Namely, the $\gamma$-polynomial equals $I_{\widetilde{G}}(4t)$.*

## Symmetric edge polytopes of type A

$G$ : a graph on $[d]$ with the edge set $E(G)$.

### Definition (Ohsugi-Hibi)

The symmetric edge polytope of type $A$ of $G$ is

$$\mathcal{A}_G := \text{conv}(\{\pm(\mathbf{e}_i - \mathbf{e}_j) : \{i, j\} \in E(G)\}).$$

### Theorem (Ohsugi-Hibi)

*$\mathcal{A}_G$ is reflexive with a regular unimodular triangulation.*

### Theorem (Higashitani-Jochemko-Michałek)

*Let $K_{m+1,n+1}$ be a complete $(m+1, n+1)$-bipartite graph. Then*

$$h^*(\mathcal{A}_{K_{m+1,n+1}}, t) = \sum_{i=0}^{\min(m,n)} \binom{2i}{i}\binom{m}{i}\binom{n}{i} t^i (t+1)^{t+t-2i+1}.$$

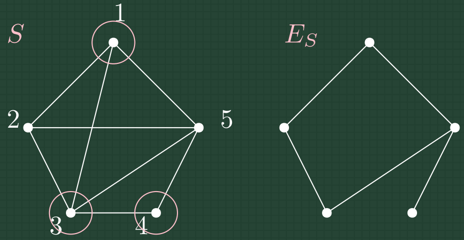
*In particular $h^*(\mathcal{A}_{K_{m+1,n+1}}, t)$ is $\gamma$-positive. (In fact, it is real-rooted.)*

## Cuts of graphs

Given a subset $S \subset [d]$,

$$E_S := \{e \in E(G) : |e \cap S| = 1\} : \text{a cut of G.}$$

We identify $E_S$ with the subgraph of $G$ on the vertex set $[d]$ and the edge set $E_S$. In particular, $E_S$ is a bipartite graph.




$\mathrm{Cut}(G)$ : the set of all cuts of $G$.

Note that $|\mathrm{Cut}(G)| = 2^{d-1}$.

## $\gamma$-positivity of $h^*(A_{\widehat{G}}, t)$

Let $\widehat{G}$ be the suspension of $G$, i.e., the connected bipartite graph on $[d+1]$ whose edge set is

$$E(\widehat{G}) = E(G) \cup \{\{i, d+1\} : i \in [d]\}.$$

**Theorem (Ohsugi-T)**

*$\mathcal{A}_{\widehat{G}}$ is unimodularly equivalent to a locally anti-blocking reflexive polytope and one has*

$$h^*(\mathcal{A}_{\widehat{G}}, t) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cut}(G)} h^*(\mathcal{B}_H, t) = (t+1)^d f_G\left(\frac{4t}{(t+1)^2}\right),$$

*where*

$$f_G(t) = \frac{1}{2^{d-1}} \sum_{H \in \mathrm{Cut}(G)} I_{\widetilde{H}}(t).$$

*Namely, the $\gamma$-polynomial equals $f_G(4t)$.*

## $\gamma$-positivity of $h^*(\mathcal{A}_{\widetilde{G}}, t)$

**Theorem (Ohsugi-T)**

*Let $G$ be a bipartite graph on $[d]$. Then the $\gamma$-polynomial of $h^*(\mathcal{A}_{\widetilde{G}}, t)$ coincides with that of $h^*(\mathcal{A}_{\widehat{G}}, t)$. Hence $h^*(\mathcal{A}_{\widetilde{G}}, t)$ is $\gamma$-positive.*

For example, by using these formulas, we can compute

$$h^*(\mathcal{A}_{K_d}, t) = \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{d-1}{2i} \binom{2i}{i} t^i (t+1)^{d-1-2i}.$$

$$h^*(\mathcal{A}_{K_{m+1,n+1}}, t) = \sum_{i=0}^{\min(m,n)} \binom{2i}{i} \binom{m}{i} \binom{n}{i} t^i (t+1)^{t+t-2i+1}.$$

## $\gamma$-positivity of $h^*(\mathcal{A}_{C_d}, t)$

Ohsugi-Shibata computed the $h^*(\mathcal{A}_{C_d}, t)$. By using the result, we can obtain

$$h^*(\mathcal{A}_{C_d}, t) = \sum_{i=0}^{\lfloor (d-1)/2 \rfloor} \binom{2i}{i} x^i (x+1)^{d-2i-1}.$$

In particular, it is $\gamma$-positive.
$\mathcal{A}_{C_{2m+1}}$ is unimodularly equivalent to the del Pezzo polytope $V_{2m}$.
We can compute the $h^*$-polynomials of pseudo-del Pezzo polytopes.

**Theorem (Ohsugi-T)**

*The $h^*$-polynomial of any pseudo-symmetric simplicial reflexive polytope is $\gamma$-positive.*

## $\gamma$-positivity of $h^*(\mathcal{A}_G, t)$

$h^*(\mathcal{A}_G, t)$ is $\gamma$-positive if one of the following

- $G = \widehat{H}$ for some graph $H$ (e.g., complete graphs, wheel graphs);
- $G = \widetilde{H}$ for some bipartite graph $H$ (e.g., complete bipartite graphs);
- $G$ is a cycle;
- $G$ is an outerplaner bipartite graph.

**Conjecture**

*$h^*(\mathcal{A}_G, t)$ is $\gamma$-positive for any graph $G$.*

## $\gamma$-positivity for locally anti-blocking reflexive polytopes

**Conjecture**

*The $h^*$-polynomial of a locally anti-blocking reflexive polytope is $\gamma$-positive.*

In order to prove this conjecture, it is enough to show the following conjecture:

**Conjecture**

*The $h^*$-polynomial of an unconditional reflexive polytope is $\gamma$-positive.*