The strong Lefschetz property of an algebra associated to a simple graphic matroid

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Outline

Matroid

E : a finite set, $\mathcal{B} \subset 2^E$

Definition

- $M = (E, \mathcal{B})$ is *matroid* with the basis $\mathcal{B}$

$\Leftrightarrow$ (B1) $\mathcal{B} \neq \emptyset$,

(B2) $B, B' \in \mathcal{B}, x \in B \setminus B' \implies \exists y \in B'$ s.t. $(B \setminus \{ x \}) \cup \{ y \} \in \mathcal{B}$.

- An element in $\mathcal{B}$ call a *base*.
- The *rank* of M is defined by the number of element of a base.

Example

Let

- $\Gamma = (V, E)$: a connected graph,
- $\mathcal{B} = \{ T \subset E \mid \text{spanning trees in } \Gamma \}$.

Then $M(\Gamma) = (E, \mathcal{B})$ is a matroid. (*graphic matroid*)

Remark

$M(\Gamma)$ is *simple* $\iff$ Γ is simple

The basis generating function

Let $M = (E, \mathcal{B})$ be a matroid with rank r.

Definition

$$F_M = \sum_{B \in \mathcal{B}} \prod_{e \in B} x_e$$

Remark

- F_M is a homogeneous polynomial of degree r.
- F_M is a sum of square-free monomials with coefficients one.
- For a graphic matroid $M(\Gamma)$, $F_{M(\Gamma)} = F_\Gamma$ is called the *Kirchhoff polynomial* of Γ.

The strong Lefschetz property

Let $R = \bigoplus_{k=0}^{r} R_k$, $R_s \neq \mathbf{0}$ be a graded Artinian ring.

Definition

R has the *strong Lefschetz property (SLP) at degree* k *with* $L \in R_1$.

$\Leftrightarrow$ the following map is bijective:

$$\begin{array}{rccc} \times L^{r-2k}\colon & R_k & \longrightarrow & R_{r-k} \\ & \Uplus & & \Uplus \\ & f & \longmapsto & L^{r-2k} \times f \end{array}$$

Definition

R has the *strong Lefschetz property (SLP) with* $L \in R_1$

$\Leftrightarrow$ $\forall k \in \left\{ 0, 1, \ldots, \lfloor \frac{r}{2} \rfloor \right\}$, R has SLP at degree k with L.

Remark

Let $\mathcal{L}_k = \{\, L \in R_1 \mid \times L^{s-2k} : R_k \to R_{s-k} \text{ is bijective} \,\}$.

$$\forall k, \mathcal{L}_k \neq \emptyset \quad \Rightarrow \quad \bigcap_k \mathcal{L}_k \neq \emptyset.$$

Proposition

- $R = \bigoplus_{k=0}^{r} R_k$: *a garded Artinian algebra,*
- $h = (h_0, h_1, \ldots, h_r)$: *the Hilbert function of* R*.*

If R *has SLP, then:*

- *the Hilbert function is symmetric* ($\forall k, h_k = h_{r-k}$),
- *the Hilbert function is unimodal.* $\nearrow \;\searrow$

Aritinian Gorenstein algebra

Let

$$F \in \mathbb{K}[x_1, x_2, \ldots, x_n] : \text{a homogeneous polynomial of degree } r,$$

$$\operatorname{Ann}(F) = \left\{ P \in \mathbb{K}[x_1, \ldots, x_n] \;\middle|\; P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) F = 0 \right\},$$

$$R = \mathbb{K}[x_1, \ldots, x_n] / \operatorname{Ann}(F) = \bigoplus_{k=0}^{r} R_k.$$

Then, R is a graded Artinian Gorenstein algebra.

SLP for Gorenstein algebras

- $R = \mathbb{K}[x_1, \ldots, x_n]/\operatorname{Ann}(F) = \bigoplus_{k=0}^{r} R_k$.
- Λ_k: $\mathbb{K}$-basis for R_k.

Definition (The k-th Hessian matrix)

$$H_F^{(k)} = \left(e_i\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) e_j\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) F\right)_{e_i, e_j \in \Lambda_k}.$$

Remark

If $\{ x_1, \ldots, x_n \}$ is a $\mathbb{K}$-basis for R_1, then $H_F^{(1)}$ is usual Hessian matrix.

Theorem (J. Watanabe, Maeno–Watanabe)

Let $L = a_1x_1 + \cdots + a_nx_n \in R_1$.

$$\times L^{r-2k} \colon R_k \longrightarrow R_{r-k} \text{ is bijective} \iff \det H_F^{(k)}(a_1, \ldots, a_n) \neq 0.$$

Main theorem

- Let $\Gamma = (V, E)$ a simple connected graph with

$$\#V = r + 1 \ (r \geq 2), \qquad E = \{ 1, 2, \dots, n \} .$$

- The Kirchhoff polynomial F_Γ is a homogeneous polynomial of degree r.
- Consider the Artinian Gorenstein algebra

$$\begin{aligned} R_\Gamma &= \mathbb{R}[x_1, \dots, x_n] / \operatorname{Ann}(F_\Gamma) \\ &= \bigoplus_{k=0}^{r} R_k \end{aligned}$$

Main theorem

Theorem (Main theorem)

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$*, we have* $\det H^{(1)}_{F_\Gamma}(\boldsymbol{a}) \neq 0$.

Moreover, the Hessian matrix $H^{(1)}_{F_\Gamma}(\boldsymbol{a})$ *has*

- *exactly one positive eigenvalue,*
- $n - 1$ *negative eigenvalues.*

By the Hessian criterion, we have the following:

Corollary

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$*, define* $L_{\boldsymbol{a}} = a_1 x_1 + \cdots + a_n x_n$*. Then*

$$\times L_{\boldsymbol{a}}^{r-2} \colon R_1 \longrightarrow R_{r-1}$$

is bijective. Therefore R_Γ *has SLP at degree* 1.

Outline

Outline of proof

- $\Gamma = (V, E)$ with $\#V = r + 1$, $E = \{ 1, 2, \ldots, n \}$,
- $R_\Gamma = \mathbb{R}[x_1, \ldots, x_n]/\operatorname{Ann}(F_\Gamma) = \bigoplus_{k=0}^{r} R_k$

Theorem (Main theorem)

For $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$, we have $\det H^{(1)}_{F_\Gamma}(\boldsymbol{a}) \neq 0$.
Moreover, the Hessian matrix $H^{(1)}_{F_\Gamma}(\boldsymbol{a})$ has

- *exactly one positive eigenvalue,*
- *$n - 1$ negative eigenvalues.*

Rough sketch

- K_r : complete graph
- Γ : subgraph of the complete graph

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$,
$\det H_{K_r}(\boldsymbol{a}) \neq 0$.

$\longrightarrow$

F_{K_r} is strictly
log-concave on $\mathbb{R}^n_{>0}$.

$\downarrow$

F_{Γ} is strictly
log-concave on $\mathbb{R}^n_{>0}$.

$\longleftarrow$

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$,
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Hessian matrices and log-concavity

Let F be a **general** homogeneous polynomial of degree r.

Definition

F is log-concave at $\boldsymbol{a} \in \mathbb{R}^n$

$$\Leftrightarrow \quad -F(\boldsymbol{x})H_F(\boldsymbol{x}) + (\operatorname{grad} F(\boldsymbol{x}))^t \cdot \operatorname{grad} F(\boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{a}}$$

is positive semidefinite.

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Hessian matrices and log-concavity

Let F be a **general** homogeneous polynomial of degree r.

Definition

F is (strictly) log-concave at $\boldsymbol{a} \in \mathbb{R}^n$

$$\Leftrightarrow \quad -F(\boldsymbol{x})H_F(\boldsymbol{x}) + (\operatorname{grad} F(\boldsymbol{x}))^t \cdot \operatorname{grad} F(\boldsymbol{x})\Big|_{\boldsymbol{x}=\boldsymbol{a}} \succeq 0 \ (\succ 0)$$

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Remark

- F is log-concave $\iff$ $\log F$ is concave $\iff$ $H_{\log F} \preceq 0$
- $$(H_{\log F})_{ij} = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\log F = \frac{F(\partial x_i \partial x_j F) - (\partial x_i F)(\partial x_i F)}{F^2}$$

Hessian matrices and log-concavity

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Lemma

Let F be a homogeneous polynimoal of degree r. Then

$$\det\left(-FH_F + (\operatorname{grad} F)^t \cdot \operatorname{grad} F\right) = (-1)^{n-1}\frac{1}{r-1}F^n \det H_F.$$

Hessian matrices and log-concavity

Let F be a **general** homogeneous polynomial of degree r.

Definition

F is (strictly) log-concave at $\boldsymbol{a} \in \mathbb{R}^n$

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Lemma

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$$\det\left(-FH_F + (\operatorname{grad} F)^t \cdot \operatorname{grad} F\right) = (-1)^{n-1}\frac{1}{r-1}F^n \det H_F.$$

Remark

F is strictly log-concave $\Longleftrightarrow$

- F is log-concave,
- $\det H_F \neq 0$.

Signature of a Hessian matrix

Theorem (Cauchy's interlacing theorem)

- $A : n \times n$ *symmetric matrix*
- $\alpha_1 \geq \cdots \geq \alpha_n$: *eigenvalues of* A.
- $B = A + \boldsymbol{v}^t \cdot \boldsymbol{v}$ $(\boldsymbol{v} \in \mathbb{R}^n)$.
- $\beta_1 \geq \cdots \geq \beta_n$: *eigenvalues of* B.

Then

$$\beta_1 \geq \alpha_1 \geq \beta_2 \geq \cdots \geq \alpha_{n-1} \geq \beta_n \geq \alpha_n.$$

Corollary

If B *is positive definite, and* $\operatorname{tr} A = 0$, *then* A *has*

- *exactly one positive eigenvalue,*
- $n - 1$ *negative eigenvalues.*

Signature of a Hessian matrix

Assume that

- F is a homogeneous polynomial in n variables,
- F is a sum of square-free monomials with positive coefficients.

Remark

- H_F is an $n \times n$ symmetric matrix.
- Each diagonal of H_F is zero. ($\Longrightarrow \operatorname{tr} H_F = 0$)

By Cauchy's interlacing theorem, we have the following:

Proposition

If F is strictly log-concave on $\mathbb{R}^n_{>0}$, then $H_F(\boldsymbol{a})(\boldsymbol{a} \in \mathbb{R}^n_{>0})$ has

- *exactly one positive eigenvalue,*
- *$n-1$ negative eigenvalues.*

Log-concavity of a Kirchhoff polynomial

Here we consider a **Kirchhoff polynomial** F_Γ.

Remark

- F_Γ is a homogeneous polynomial.
- F_Γ is a sum of square-free monomials with coefficients one.

Theorem (Anari–Oveis Gharan–Vinzant)

For any martoid M, the basis genarating function F_M is log-concave on $\mathbb{R}^n_{>0}$. In particular, a Kirchhoff polynomial is log-concave.

By this theorem, we have the following:

Remark

For a Kirchhoff polynomial F_Γ,
F_Γ is strictly log-concave on $\mathbb{R}^n_{>0} \iff \det H_{F_\Gamma}(\boldsymbol{a}) \neq 0$ ($\boldsymbol{a} \in \mathbb{R}^n_{>0}$).

Rough sketch

- K_r : complete graph
- Γ : subgraph of the complete graph

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$,
$\det H_{K_r}(\boldsymbol{a}) \neq 0$.

$\longrightarrow$

F_{K_r} is strictly
log-concave on $\mathbb{R}^n_{>0}$.

$\downarrow$

F_Γ is strictly
log-concave on $\mathbb{R}^n_{>0}$.

$\longleftarrow$

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$,
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- K_r : complete graph
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For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, det $H_{K_r}(\boldsymbol{a}) \neq 0$.	$\longrightarrow$	F_{K_r} is strictly log-concave on $\mathbb{R}^n_{>0}$.
		$\downarrow$
For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, det $H_{\Gamma}(\boldsymbol{a}) \neq 0$.	$\longleftarrow$	F_{Γ} is strictly log-concave on $\mathbb{R}^n_{>0}$.

Assume that

- $F \in \mathbb{R}[x_1, \dots, x_n]$ is a homogeneous polynomial of degree r,
- F is a sum of square-free monomials with positive coefficients.

Lemma ($\star$)

Assume that F is strictly log-concave on $\mathbb{R}^n_{>0}$. If

$$\frac{\partial F}{\partial x_1} \neq 0, \frac{\partial F|_{x_1=0}}{\partial x_2} \neq 0, \dots, \frac{\partial F|_{x_1=\cdots=x_{k-1}=0}}{\partial x_k} \neq 0$$

holds for some $0 \leq k \leq n-r$,

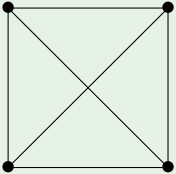
then $F|_{x_1=\cdots=x_k=0} \in \mathbb{R}[x_{k+1}, \dots, x_n]$ is strictly log-concave on $\mathbb{R}^{n-k}_{>0}$.

Lemma

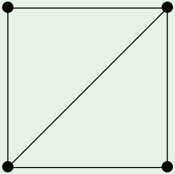
Every Kirchhoff polynomial is obtained from the Kirchhoff polynomial of the complete graph with same number vertices by substituting zero for some variables.

Example

K_4 :



$K_4 \setminus e$:



$$F_{K_4 \setminus e} = F_{K_4}|_{x_e=0}.$$

We can apply the **Kirchhoff polynomial** F_{K_r} to Lemma($\star$)

Lemma ($\star$)

Assume that F is strictly log-concave on $\mathbb{R}^n_{>0}$. If

$$\frac{\partial F}{\partial x_1} \not\equiv 0, \frac{\partial F|_{x_1=0}}{\partial x_2} \not\equiv 0, \ldots, \frac{\partial F|_{x_1=\cdots=x_{k-1}=0}}{\partial x_k} \not\equiv 0$$

holds for some $0 \leq k \leq n-r$,

then $F|_{x_1=\cdots=x_k=0} \in \mathbb{R}[x_{k+1}, \ldots, x_n]$ is strictly log-concave on $\mathbb{R}^{n-k}_{>0}$.

Rough sketch

- K_r : complete graph
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For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, det $H_{K_r}(\boldsymbol{a}) \neq 0$. $\longrightarrow$ F_{K_r} is strictly log-concave on $\mathbb{R}^n_{>0}$.

$\downarrow$

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, det $H_{\Gamma}(\boldsymbol{a}) \neq 0$. $\longleftarrow$ F_{Γ} is strictly log-concave on $\mathbb{R}^n_{>0}$.

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For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, $\det H_{K_r}(\boldsymbol{a}) \neq 0$. $\longrightarrow$ F_{K_r} is strictly log-concave on $\mathbb{R}^n_{>0}$.

$\downarrow$

For $\boldsymbol{a} \in \mathbb{R}^n_{>0}$, $\det H_{\Gamma}(\boldsymbol{a}) \neq 0$. $\longleftarrow$ F_{Γ} is strictly log-concave on $\mathbb{R}^n_{>0}$.

Proposition

For $r \geq 3$,

$$\det H^{(1)}_{F_{K_r}}(\boldsymbol{x}) = (-1)^{\binom{r}{2}-1} 2^{\binom{r}{2}-r-1}(r-2)(F_{K_r}(\boldsymbol{x}))^{\binom{r}{2}-r}.$$

Remark

- It is known that $\det H^{(1)}_{F_{K_r}}(1,\ldots,1) \neq 0$.
- Since the Kirchhoff polynomial is a sum of monomials with positive coefficients, Proposition implies that

$$\det H^{(1)}_{K_r}(\boldsymbol{a}) \neq 0, (\boldsymbol{a} \in \mathbb{R}^n_{>0}).$$

Hessians and Prehomogeneous vector spaces

(G, ρ, V) : a prehomogeneous vector space/$\mathbb{C}$

Definition

$F \in \mathbb{C}(V)$ is a *relative invariant* (corresponding to χ)

$\Leftrightarrow$ $\exists \chi \in \mathrm{Hom}(G \to \mathbb{C}^*)$ s.t. $\forall g \in G, \forall \boldsymbol{x} \in V,\ F(\rho(g)\boldsymbol{x}) = \chi(g)F(\boldsymbol{x})$.

Proposition

1. $F \in C(V)$ *is a relative invariant*
   $\Longrightarrow$ $\det H_F$ *is also a relative invariant.*
2. (G, ρ, V)*: irreducible prehomo.* ($\Longleftrightarrow$ ρ *: irreducible*)
   $\Longrightarrow$ *Then there is at most one irreducible relative invariant F up to constant multiple. In particular, any relative invariant is in the form of cF^m for $c \in \mathbb{C}$ and $m \in \mathbb{Z}$.*

By previous proposition, we have the following:

Corollary

- (G, ρ, V) : *irreducible prehomogeneous vector space,*
- F : *irreducible relative invariant.*

$\Longrightarrow \ \exists c, \exists m, \ \ s.t. \quad \det H_F = cF^m$

Presentation of Kirchhoff polynomials

- Let $\Gamma = (V, E)$ be a simple graph
- $e = \{ i, j \} \in E \ \longleftrightarrow \ x_e$
- For $e = \{ i, j \} \in E$, define

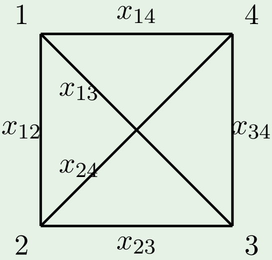
$$l_{ij} = \begin{cases} \sum_{k \sim i} x_{ik} & i = j, \\ -x_{ij} & i \sim j, \\ 0 & i \not\sim j. \end{cases}$$

- $L_\Gamma = (l_{ij})_{i,j \in V}$ (the *Laplacian* of Γ)

Theorem (The matrix-tree theorem)

$$\forall i, j, \ F_\Gamma = \det L_\Gamma^{(ij)}.$$

Example



$$L_{K_4} = \begin{bmatrix} x_{12}+x_{13}+x_{14} & -x_{12} & -x_{13} & -x_{14} \\ -x_{12} & x_{12}+x_{23}+x_{24} & -x_{23} & -x_{24} \\ -x_{13} & -x_{23} & x_{13}+x_{23}+x_{34} & -x_{34} \\ -x_{14} & -x_{24} & -x_{34} & x_{14}+x_{24}+x_{34} \end{bmatrix}$$

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Example

$$L_{K_4}^{(11)} = \begin{bmatrix} x_{12}+x_{13}+x_{14} & -x_{12} & -x_{13} & -x_{14} \\ -x_{12} & x_{12}+x_{23}+x_{24} & -x_{23} & -x_{24} \\ -x_{13} & -x_{23} & x_{13}+x_{23}+x_{34} & -x_{34} \\ -x_{14} & -x_{24} & -x_{34} & x_{14}+x_{24}+x_{34} \end{bmatrix}$$

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$$\therefore \left\{ L_{K_4}^{(11)} \,\middle|\, x_{ij} \in \mathbb{C} \right\} = \{\, 3 \times 3 \text{ symmetric matrix} \,\} =: \mathrm{Sym}(3, \mathbb{C}).$$

By $F_{K_4} = \det L_{K_4}^{(11)}$, we can regard $F_{K_4} : \mathrm{Sym}(3, \mathbb{C}) \to \mathbb{C}$.

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$$\therefore \left\{ L_{K_4}^{(11)} \,\middle|\, x_{ij} \in \mathbb{C} \right\} = \{ 3 \times 3 \text{ symmetric matrix} \} =: \mathrm{Sym}(3, \mathbb{C}).$$

By $F_{K_4} = \det L_{K_4}^{(11)}$, we can regard $F_{K_4} : \mathrm{Sym}(3, \mathbb{C}) \to \mathbb{C}$.

Proposition

- *In general,* $\left\{ L_{K_{r+1}}^{(11)} \,\middle|\, x_{ij} \in \mathbb{C} \right\} = \mathrm{Sym}(r, \mathbb{C})$,
- *Hence we can regard* $F_{K_{r+1}} : \mathrm{Sym}(r, \mathbb{C}) \to \mathbb{C}$.

The Hessian of the Kirchhoff polynomial of K_{r+1}

Define

$$\begin{array}{rccc} \rho : & \mathrm{GL}_r(\mathbb{C}) & \to & \mathrm{GL}(\mathrm{Sym}(r,\mathbb{C})) \\ & P & \mapsto & \left(\begin{array}{rccc} \rho(P): & \mathrm{Sym}(n,\mathbb{C}) & \to & \mathrm{Sym}(r,\mathbb{C}) \\ & X & \mapsto & PXP^t \end{array} \right). \end{array}$$

Proposition (cf. Kimura–Sato)

Then $(\mathrm{GL}_r(\mathbb{C}), \rho, \mathrm{Sym}(r,\mathbb{C}))$ *is an irreducible prehomogeneous vector space. Moreover,* $\det : \mathrm{Sym}(r,\mathbb{C}) \to \mathbb{C}$ *is an irreducible relative invariant.*

Proposition

The Kirchhoff polynomial F_{K_r} is an irreducible relative invariant.

Proposition (Y.)

$$\det H_{K_r}^{(1)}(1,\ldots,1) = (-1)^{\binom{r}{2}-1} 2^{\binom{r}{2}-r+1} r^{r+\binom{r}{2}(r-4)} (r-2) \neq 0.$$

Combining these propositions, we have the following:

Theorem

$$\det H_{K_r}^{(1)} = (-1)^{\binom{r}{2}-1} 2^{\binom{r}{2}-r-1} (r-2) (F_{K_r})^{\binom{r}{2}-r}$$