符号から得られる多項式とデザイン理論の関係

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Definition (t-(n, k, λ) design)

Let

- $X := \{1, 2, \ldots, n\}$
- $\mathcal{B} \subseteq \binom{X}{k}$

We say $(X, \mathcal{B})$ is a t-(n, k, λ) design, if there exists λ such that for all $T \in \binom{X}{t}$, $\lambda = |\{B \in \mathcal{B} \mid T \subseteq B\}|$.

Combinatorial Designs

Example

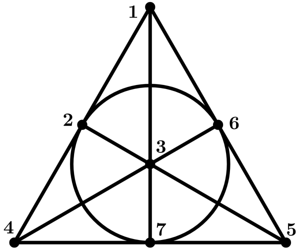
Let

$$\begin{cases} X = \{1,2,3,4,5,6,7\} \\ \mathcal{B} = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\} \end{cases}$$

Then,

$$\forall T \in \binom{X}{2}, |\{B \in \mathcal{B} \mid T \subseteq B\}| = 1.$$

$(X, \mathcal{B})$ is a 2-$(7,3,1)$ design.



Codes⇒Combinatorial Designs

Problem

How do we obtain a t-designs structure?

Let C be an $\mathbb{F}_q$-linear code of length n. For $c = (c_1, c_2, \ldots, c_n) \in C$,

- $\mathrm{supp}(c) := \{i \mid c_i \neq 0\}$
- $\mathrm{wt}(c) := |\,\mathrm{supp}(c)|$
- $C_k := \{c \in C \mid \mathrm{wt}(c) = k\}$

Definition (t-design obtained from a code)

Let C be an $\mathbb{F}_q$-linear code of length n.

$$\begin{cases} X = \{1, 2, \ldots, n\} \\ \mathcal{B}(C_k) = \{\mathrm{supp}(c) \mid c \in C_k\} \subseteq \binom{X}{k} \end{cases}$$

If $(X, \mathcal{B}(C_k))$ is a t-(n, k, λ) design, then C_k is a t-(n, k, λ) design.

We say C is a t-homogeneous if C_k is a t-design for all $k (C_k \neq \emptyset)$.

Codes⇒Combinatorial Designs

Example (Hamming code)

Let H be the Hamming $[7,4]$ code (4-dim subspace of $\mathbb{F}_2^7$) with the following generator matrix G:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$H_3 = \{(1,1,0,1,0,0,0),(0,1,1,0,1,0,0),(0,0,1,1,0,1,0),$
$(0,0,0,1,1,0,1),(1,0,0,0,1,1,0),(0,1,0,0,0,1,1),(1,0,1,0,0,0,1)\}$,

$$\begin{cases} X = \{1,2,3,4,5,6,7\} \\ \mathcal{B}(H_3) = \{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,7\},\{5,6,1\},\{6,7,2\},\{7,1,3\}\} \end{cases}$$

Then,

$$\forall T \in \binom{X}{2}, |\{B \in \mathcal{B} \mid T \subseteq B\}| = 1.$$

$(X, \mathcal{B}(H_3))$ is a 2-$(7,3,1)$ design. $\Rightarrow$ H_3 is a 2-$(7,3,1)$ design.

Codes⇒Combinatorial Designs

We can obtain a t-design from a code by using the following:

- Harmonic weight enumerators
- Jacobi polynomials (Main)

Designs⇔Harmonic weight enumerators

Definition (Discrete harmonic function)

Let

- $\Omega = \{1, \ldots, n\}$, $X = 2^{\Omega}$, $X_k = \binom{X}{k}$
- $\mathbb{R}X = \{\sum_{x \in X} c_i x \mid \forall i, c_i \in \mathbb{R}\}$
- $\mathbb{R}X_k = \{\sum_{x \in X_k} c_i x \mid \forall i, c_i \in \mathbb{R}\}$ $(k = 0, 1, \ldots, n)$

For $f \in \mathbb{R}X_k$, f is denoted by

$$f = \sum_{z \in X_k} f(z) z$$

and we can identify f with a function on X_k given by $z \to f(z)$.
Moreover, f can be extend to $\widetilde{f} \in \mathbb{R}X$: for $u \in X$,

$$\widetilde{f}(u) = \sum_{z \in X_k, z \subset u} f(z).$$

Designs⇔Harmonic weight enumerators

- Let $\gamma : \mathbb{R}X_k \to \mathbb{R}X_{k-1}$ be a map defined by linearity from
$$\gamma(z) = \sum_{y \in X_{k-1}, y \subset z} y \text{ for all } z \in X_k.$$
- $\mathrm{Harm}_k = \ker(\gamma|_{\mathbb{R}X_k})$.

Then, an element of Harm_k is a harmonic function.

Example

Let $\Omega = \{1, 2, 3, 4\}$ and

$$f = \{1, 2\} + \{1, 3\} - 2\{1, 4\} - 2\{2, 3\} + \{2, 4\} + \{3, 4\} \in \mathbb{R}X_2.$$

Then

$$\begin{aligned}\widetilde{f}(\{1, 2, 3\}) &= f(\{1, 2\}) + f(\{1, 3\}) + f(\{2, 3\}) = 1 + 1 - 2 = 0,\\ \gamma(f) =&(\{1\} + \{2\}) + (\{1\} + \{3\})\\ &- 2(\{1\} + \{4\}) - 2(\{2\} + \{3\})\\ &+ (\{2\} + \{4\}) + (\{3\} + \{4\}) = 0\end{aligned}$$

and $f \in \mathrm{Harm}_2$.

Designs⇔Harmonic weight enumerators

Theorem (Delsarte ,1978)

Let C be a binary code of length n.
C_ℓ is a t-design $\Leftrightarrow$
$\sum_{c \in C_\ell} \widetilde{f}(\mathrm{supp}(c)) = 0, \forall f \in \mathrm{Harm}_k \ (1 \leq k \leq t)$

Example

Let

- C be a binary code of length 4.
- $C_2 = \{(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,1,0),(0,1,0,1),(0,0,1,1)\}$.
- $\mathcal{B}(C_2) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.

Then, C_2 is a 2-$(4,2,1)$ design.
$\Leftrightarrow$
$f = \{1,2\} + \{1,3\} - 2\{1,4\} - 2\{2,3\} + \{2,4\} + \{3,4\} \in \mathrm{Harm}_2$.

$$\sum_{c \in C_2} \widetilde{f}(\mathrm{supp}(c)) = 1 + 1 - 2 - 2 + 1 + 1 = 0.$$

Designs⇔Harmonic weight enumerators

Definition (Harmonic weight enumerator (Bachoc, 1999))

A harmonic weight enumerator of C with $f \in \mathrm{Harm}_k$ is defined as follows:

$$w_{C,f}(x,y) = \sum_{c \in C} \widetilde{f}(\mathrm{supp}(c)) x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)}.$$

Example (Hamming code)

Let H be the Hamming $[7,4]$ code (4-dim subspace of $\mathbb{F}_2^7$) with the following generator matrix G:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$H = \{(0,0,0,0,0,0,0), (1,1,0,1,0,0,0), (0,1,1,0,1,0,0), \ldots\},$$

Then,

$$w_{H,f}(x,y) = \widetilde{f}(\emptyset) x^{7-0} y^0 + \widetilde{f}(\{1,2,4\}) x^{7-3} y^3 + \widetilde{f}(\{2,3,5\}) x^{7-3} y^3 + \cdots.$$

Designs⇔Harmonic weight enumerators

Using

$$\begin{aligned} w_{C,f}(x,y) &= \sum_{c \in C} \widetilde{f}(\mathrm{supp}(c)) x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)} \\ &= \sum_{\ell=0}^{n} \Big(\sum_{c \in C_\ell} \widetilde{f}(\mathrm{supp}(c)) \Big) x^{n-\ell} y^{\ell}, \end{aligned}$$

we have the following theorem.

Theorem

If for all $f \in \mathrm{Harm}_k$ $(1 \leq k \leq t)$, $w_{C,f}(x,y) = 0$, then, for all ℓ, C_ℓ $(\neq \emptyset)$ is a t-design.

Designs⇔Harmonic weight enumerators

Example (Hamming code)

$$
\begin{aligned}
w_{H,f}(x,y) &= \widetilde{f}(\emptyset)x^7 + \widetilde{f}(\{1,2,4\})x^4y^3 + \widetilde{f}(\{2,3,5\})x^4y^3 + \cdots \\
&= 0 + \left(f(\{1,2\}) + f(\{1,4\}) + f(\{2,4\})\right) x^4y^3 \\
&\quad + \left(f(\{2,3\}) + f(\{2,5\}) + f(\{3,5\})\right) x^4y^3 \\
&\quad + \left(f(\{1,3\}) + f(\{1,7\}) + f(\{3,7\})\right) x^4y^3 \\
&\quad + \left(f(\{1,5\}) + f(\{1,6\}) + f(\{5,6\})\right) x^4y^3 + \cdots \\
&= 0 + (1+1+1)x^4y^3 + (1+1+1)x^4y^3 \\
&\quad + (1-5-5)x^4y^3 + (1+1-14)x^4y^3 + \cdots \\
&= 0.
\end{aligned}
$$

$(f = \{1,2\} + \{1,3\} + \{1,4\} + \{1,5\} + \{1,6\} - 5\{1,7\} \cdots + 10\{6,7\} \in \mathrm{Harm}_2)$

- H_3 is a 2-design.
- H_4 is a 2-design.

Codes⇒Combinatorial Designs

We can obtain a t-design from a code by using the following:

- Harmonic weight enumerators
- Jacobi polynomials (Main)

Designs⇔Jacobi polynomials

Let C be an $\mathbb{F}_q$-linear code of length n.

Definition (Weight enumerator)

$$W_C(x, y) := \sum_{c \in C} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)}.$$

Definition (Jacobi polynomial (M. Ozeki, 1997))

Let $T \subseteq \{1, \ldots, n\} = [n]$.

$$J_{C,T}(w, z, x, y) := \sum_{c \in C} w^{m_0(c)} z^{m_1(c)} x^{n_0(c)} y^{n_1(c)},$$

where

- $m_0(c) = |\{j \in T \mid c_j = 0\}|$
- $m_1(c) = |\{j \in T \mid c_j \neq 0\}|$
- $n_0(c) = |\{j \in [n] \setminus T \mid c_j = 0\}|$
- $n_1(c) = |\{j \in [n] \setminus T \mid c_j \neq 0\}|$

Designs⇔Jacobi polynomials

Example (Hamming code)

Let H be the Hamming $[7,4]$ code (4-dim subspace of $\mathbb{F}_2^7$) with the following generator matrix G:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$$H = \{(0,0,0,0,0,0,0),(1,1,0,1,0,0,0),(0,1,1,0,1,0,0),\\ (0,0,1,1,0,1,0),(0,0,0,1,1,0,1),\dots,(1,1,1,1,1,1,1)\},$$

Then,

$$\begin{aligned} W_H(x,y) &= x^{7-0}y^0 + x^{7-3}y^3 + \cdots + x^{7-7}y^7 \\ &= x^7 + 7x^4y^3 + 7x^3y^4 + y^7. \end{aligned}$$

Designs⇔Jacobi polynomials

Example (Hamming code)

$$H = \{(0,0,0,0,0,0,0),(1,1,0,1,0,0,0),(0,1,1,0,1,0,0),\\ (0,0,1,1,0,1,0),(0,0,0,1,1,0,1),\ldots,(1,1,1,1,1,1,1)\},$$

H_3 is a 2-$(7,3,1)$ design.
If $T = \{1,2\}$, then

$$J_{H,\{1,2\}}(w,z,x,y)$$
$$= w^2 z^0 x^5 y^0 + w^0 z^2 x^4 y^1 + \cdots + w^0 z^2 x^0 y^5$$
$$= w^2(x^5 + 2x^2y^3 + xy^4) + wz(4x^3y^2 + 4x^2y^3) + z^2(1x^4y + 2x^3y^2 + y^5).$$

The coefficient of $z^t x^{n-k} y^{k-t}$ in $J_{C,T} = |\{c \in C_k \mid T \subset \mathrm{supp}(c)\}|$.

Theorem

If C is a t-homogeneous, then $J_{C,T}$ is uniquely determined for all $T \subset [n]$ $(|T| = t)$.

This Jacobi polynomial is denoted by $J_{C,t}$.

Designs⇔Jacobi polynomials

Example (Hamming code)

Let H be the Hamming $[7,4]$ code (4-dim subspace of $\mathbb{F}_2^7$) with the following generator matrix G:

$$G = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

- H_3 is a 2-$(7,3,1)$design.
- H_4 is a 2-$(7,4,2)$design.

$\Rightarrow$ H is a 2-homogeneous.

$$J_{H,1}(w,z,x,y)$$
$$= w(x^6 + 4x^3y^3 + 3x^2y^4) + z(3x^4y^2 + 4x^3y^3 + y^6),$$
$$J_{H,2}(w,z,x,y)$$
$$= w^2(x^5 + 2x^2y^3 + xy^4) + wz(4x^3y^2 + 4x^2y^3) + z^2(x^4y + 2x^3y^2 + y^5).$$

Contents

Reference

- H. S. Chakraborty, T. Miezaki, M. Oura, Y. Tanaka. : Jacobi polynomials and designs theory I, *Discrete Mathematics*, **346** (2023) no. 6, No. 113339.

Extended t-design

Definition (t-$(n, k, \lambda_1^{a_1}, \ldots, \lambda_N^{a_N})$ design)

Let

- $X := \{1, 2, \ldots, n\}$
- $\mathcal{B} \subseteq \binom{X}{k}$

We say $(X, \mathcal{B})$ is a t-$(n, k, \lambda_1^{a_1}, \ldots, \lambda_N^{a_N})$ design, if there exists some $\lambda_1, \ldots, \lambda_N$, where

$$a_i = \left|\left\{T \in \binom{X}{t} \mid \lambda_i = |\{B \in \mathcal{B} \mid T \subseteq B\}|\right\}\right| \quad (a_i > 0).$$

Remark

If $\lambda_1 \leq \ldots \leq \lambda_N$, then $(X, \mathcal{B})$ is a

- t-(n, k, λ_N) packing design,
- t-(n, k, λ_1) covering design.

Moreover, the maximum (resp. minimum) number of blocks is denoted by $D_{\lambda_N}(n, k, t)$ (resp. $C_{\lambda_1}(n, k, t)$).

Codes⇒Combinatorial Designs

Definition (t-design obtained from a code)

Let C be an $\mathbb{F}_q$-linear code of length n.

$$\begin{cases} X = \{1, 2, \ldots, n\}, \\ \mathcal{B}(C_k) = \{\mathrm{supp}(c) \mid c \in C_k\} \subseteq \binom{X}{k} \end{cases}$$

If $(X, \mathcal{B}(C_k))$ is a t-$(n, k, \lambda_1^{a_1}, \ldots, \lambda_N^{a_N})$ design, then C_k is a t-$(n, k, \lambda_1^{a_1}, \ldots, \lambda_N^{a_N})$ design, where

$$a_i = \left|\left\{T \in \binom{X}{t} \mid \lambda_i = |\{B \in \mathcal{B}(C_k) \mid T \subseteq B\}|\right\}\right| \quad (\lambda_i > 0,\ a_i > 0).$$

When $N = 1$, C_k is a t-design.

Definition (Type III code)

A self-dual code C over $\mathbb{F}_3$ of length $n \equiv 0 \pmod 4$ is called *Type III* if the weight of each codeword of C is a multiple of 3.

Codes⇒Combinatorial Designs

Example (Type III code of length 8 C_{III^8} (unique))

$$
\begin{aligned}
X &= \{1,2,3,4,5,6,7,8\},\\
(C_{\mathrm{III}}^8)_6 &= \{(1,2,2,0,0,2,2,2),(0,1,1,1,1,2,2,0),(1,1,2,1,\\
&\quad 2,0,0,1),(1,2,0,1,0,2,1,2),(2,0,2,2,2,2,1,0),\ldots\},\\
\mathcal{B}((C_{\mathrm{III}}^8)_6) &= \{\{1,2,3,6,7,8\},\{2,4,5,6,7,8\},\{2,3,4,5,6,7\},\\
&\quad \{1,2,3,4,5,7\},\{1,2,3,4,5,8\},\{2,3,5,6,7,8\},\\
&\quad \{1,2,4,6,7,8\},\{2,3,4,5,6,8\},\{1,2,3,5,7,8\},\\
&\quad \{1,3,4,5,6,8\},\{1,4,5,6,7,8\},\{1,3,5,6,7,8\},\\
&\quad \{1,2,3,4,6,8\},\{1,2,3,4,6,7\},\{1,2,4,5,7,8\},\\
&\quad \{1,3,4,5,6,7\}\},
\end{aligned}
$$

Then,

$$
\begin{aligned}
a_1 &= \left|\left\{T \in \binom{X}{2} \;\middle|\; |\{B \in \mathcal{B}((C_{\mathrm{III}}^8)_6) \mid T \subseteq B\}| = 8\right\}\right|\\
&= |\{\{1,2\},\{1,5\},\{1,6\},\ldots\}| = 12,
\end{aligned}
$$

Codes⇒Combinatorial Designs

Example (Type III code of length 8 C_{III^8} (unique))

$$\begin{aligned}\mathcal{B}((C_{\mathrm{III}}^8)_6) = \{&\{1,2,3,6,7,8\},\{2,4,5,6,7,8\},\{2,3,4,5,6,7\},\\ &\{1,2,3,4,5,7\},\{1,2,3,4,5,8\},\{2,3,5,6,7,8\},\\ &\{1,2,4,6,7,8\},\{2,3,4,5,6,8\},\{1,2,3,5,7,8\},\\ &\{1,3,4,5,6,8\},\{1,4,5,6,7,8\},\{1,3,5,6,7,8\},\\ &\{1,2,3,4,6,8\},\{1,2,3,4,6,7\},\{1,2,4,5,7,8\},\\ &\{1,3,4,5,6,7\}\},\end{aligned}$$

Then,

$$\begin{aligned} a_2 &= \left|\left\{T \in \binom{X}{2} \mid |\{B \in \mathcal{B}((C_{\mathrm{III}}^8)_6) \mid T \subseteq B\}| = 9\right\}\right| \\ &= |\{\{4,5\},\{1,3\},\{1,4\},\ldots\}| = 16.\end{aligned}$$

Then, $(X, \mathcal{B}((C_{\mathrm{III}}^8)_6))$ is a 2-$(8,6,8^{12},9^{16})$ design.
Therefore, $(C_{\mathrm{III}}^8)_6$ is a 2-$(8,6,8^{12},9^{16})$ design.

- 2-$(8,6,9)$ packing design
- 2-$(8,6,8)$ covering design

Designs⇔Jacobi polynomials

Theorem

For any $T \subseteq [n]$ ($|T| = t$), there are precisely N members the coefficient of the term $w^{i_1} z^{i_2} x^{j_1} y^{j_2}$ ($i_2 + j_2 = k$) of Jacobi polynomials. Then, C_k is a t-$(n, k, \lambda_1^{a_1}, \ldots, \lambda_N^{a_N})$ design.

Example (Type II code of length 16)

Let C be the first Type II code of length 16 in "M. Harada and A. Munemasa, Database of self-dual codes".

C_{12} is a 2-$(16, 12, (15^{112}, 21^{8}))$ design.
⇔

$$J^1_{C,2} = w^2(x^{14} + 15x^{10}y^4 + 47x^6y^8 + x^2y^{12}) + wz(12x^{11}y^3 + 104x^7y^7 + 12x^3y^{11}) + z^2(x^{12}y^2 + 47x^8y^6 + 15x^4y^{10} + y^{14}),$$

$$J^2_{C,2} = w^2(x^{14} + 21x^{10}y^4 + 35x^6y^8 + 7x^2y^{12}) + 128wzx^7y^7 + z^2(7x^{12}y^2 + 35x^8y^6 + 21x^4y^{10} + y^{14}).$$

Designs⇔Jacobi polynomials

The number of the basis of $J_{C,T}$ $\Rightarrow$ We use "Molien series".

Theorem

Jacobi polynomial $J_{C^n_{\mathrm{III}},T}$ is invariant under the action of group

$$G_3 = \left\langle \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{2\pi i/3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{2\pi i/3} \end{bmatrix} \right\rangle$$

of order 48.

Designs⇔Jacobi polynomials

Definition (Molien series)

$$M_{G_3}(u,v) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \dim M_{i,n-i} u^i v^{n-i}$$

is called *Molien series*, where

$$M_{i,n-i} := (\mathbb{C}[w,z,x,y]^{G_3})_{i,n-i},$$

$$(\mathbb{C}[w,z,x,y]^{G_3})_{i,n-i} = \{f \in \mathbb{C}[w,z,x,y] \mid \forall g \in G_3,\ gf = f,$$

degree of w, z in f is i,

degree of x, y in f is $n-i\}$.

Theorem (R. P. Stanley, 1979)

$$M_{G_3}(u,v) = \frac{1}{|G_3|}\sum_{g\in G_3}\frac{1}{\det(1-ug)\det(1-vg)}.$$

Designs⇔Jacobi polynomials

Example (Type III code of length 8 C^8_{III} (unique))

The homogeneous part of degree 8 of the Molien series $M_{G_3}(u,v)$ is

$$u^8 + u^7 v + 2u^6v^2 + 2u^5v^3 + 2u^4v^4 + 2u^3v^5 + 2u^2v^6 + uv^7 + v^8.$$

Then, $M_{2,6}$ is generated by 2 polynomials as follows:

$$M_{2,6} = \langle W_{C^4_{\mathrm{III}}}(J_{C^4_{\mathrm{III}},2}),\ (J_{C^4_{\mathrm{III}},1})^2 \rangle$$

The coefficient of each basis is determined by comparing the terms of Jacobi polynomials.

$$J^1_{C^8_{\mathrm{III}},2} = 1 \cdot W_{C^4_{\mathrm{III}}}(J_{C^4_{\mathrm{III}},2}) + 0 \cdot (J_{C^4_{\mathrm{III}},1})^2$$
$$J^2_{C^8_{\mathrm{III}},2} = 0 \cdot W_{C^4_{\mathrm{III}}}(J_{C^4_{\mathrm{III}},2}) + 1 \cdot (J_{C^4_{\mathrm{III}},1})^2$$

We can obtain Jacobi polynomials as follows:

$$J^1_{C^8_{\mathrm{III}},2} = w^2(x^6 + 8x^3y^3) + wz(4x^4y^2 + 32xy^5) + z^2(4x^5y + 32x^2y^4),$$
$$J^2_{C^8_{\mathrm{III}},2} = w^2(4x^3y^3 + 4y^6) + wz(12x^4y^2 + 24xy^5) + 36z^2x^2y^4.$$

Designs⇔Jacobi polynomials

$$J^1_{C^8_{\mathrm{III}},2} = w^2(x^6+8x^3y^3)+wz(4x^4y^2+32xy^5)+z^2(4x^5y+32x^2y^{6-2}),$$
$$J^2_{C^8_{\mathrm{III}},2} = w^2(4x^3y^3+4y^6)+wz(12x^4y^2+24xy^5)+z^2(36x^2y^{6-2}).$$

$(C^8_{\mathrm{III}})_6$ is a 2-$(8,6,(8^{12},9^{16}))$ design.

$$\Rightarrow \lambda_1 = 8 = 1/4 \times (\text{the coefficient of the term} z^2x^2y^{6-2} \text{ of } J^1_{C^8_{\mathrm{III}},2})$$
$$= 1/4 \times 32,$$
$$\lambda_2 = 9 = 1/4 \times (\text{the coefficient of the term} z^2x^2y^{6-2} \text{ of } J^2_{C^8_{\mathrm{III}},2})$$
$$= 1/4 \times 36.$$

III	2-$(8,6,(8^{12},9^{16}))$
	4-$(16,9,(88^{28},92^{336},94^{896},96^{560}))$
	2-$(20,9,(392^{90},432^{100}))$
	2-$(20,12,(4212^{100},4296^{90}))$
	2-$(20,15,(6308^{90},6444^{100}))$
IV	2-$(6,4,(1^{12},2^{3}))$

Ex:Type III code of length 8 C^8_{III} (unique)

$$W_{C^8_{\text{III}}}(x,y) = x^8 + 16x^5y^3 + 64x^2y^6.$$

$(C^8_{\text{III}})_6$ is a 2-$(8,6,8)$ packing design and a 2-$(8,6,9)$ covering design.

$$\begin{aligned} C_8(8,6,2) &\leq 16 \leq D_9(8,6,2) \\ \Rightarrow 16 &= 1/4 \times (\text{the coefficient of the term} x^2y^6 \text{ of } W_{C^8_{\text{III}}}) \\ &= 1/4 \times 64, \end{aligned}$$

III	$C_8(8,6,2) \leq 16 \leq D_9(8,6,2)$
	$C_{88}(16,9,4) \leq 1360 \leq D_{96}(16,9,4)$
	$C_{392}(20,9,2) \leq 2180 \leq D_{432}(20,9,2)$
	$C_{4212}(20,12,2) \leq 12240 \leq D_{4296}(20,12,2)$
	$C_{6308}(20,15,2) \leq 11544 \leq D_{6444}(20,15,2)$
IV	$C_1(6,4,2) \leq 3 \leq D_2(6,4,2)$

Contents

Reference

- H. S. Chakraborty, T. Miezaki, M. Oura, Y. Tanaka. : Jacobi polynomials and designs theory I, *Discrete Mathematics*, **346** (2023) no. 6, No. 113339.

MacWilliams type identity

Theorem (M. Ozeki, 1997)

If C is a binary linear code, then

$$J_{C^\perp,T}(w,z,x,y) = \frac{1}{|C|}(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes I) J_{C,T}(w,z,x,y).$$

Example (Hamming code)

Let $H^\perp$ be the dual code of H.
Then,

$$\begin{aligned} J_{H^\perp,1}(w,z,x,y) &= w(x^6 + 3x^2y^4) + 4zx^3y^3 \\ &= J_{H,1}(w+z, w-z, x+y, x-y), \\ J_{H^\perp,2}(w,z,x,y) &= w^2(x^5 + xy^4) + 4wzx^2y^3 + 2z^2x^3y^2 \\ &= J_{H,2}(w+z, w-z, x+y, x-y). \end{aligned}$$

※ $J_{H,1}(w,z,x,y) = w(x^6 + 4x^3y^3 + 3x^2y^4) + z(3x^4y^2 + 4x^3y^3 + y^6)$,
$J_{H,2}(w,z,x,y) = w^2(x^5 + 2x^2y^3 + xy^4) + wz(4x^3y^2 + 4x^2y^3) + z^2(x^4y + 2x^3y^2 + y^5)$.

Jacobi polynomial with respect to ℓ reference vectors

Definition (Jacobi polynomial)

Let C be an $\mathbb{F}_q$-linear code of length n. Let $\mathbf{T} := (T_1, \ldots, T_\ell)$ of pairwise disjoint sets $T_i \subseteq [n]$.

$$J_{C,\mathbf{T}}(w_1, z_1, \ldots, w_\ell, z_\ell, x_0, x_1) := \sum_{c \in C} w_1^{m_0^1(c)} z_1^{m_1^1(c)} \cdots w_\ell^{m_0^\ell(c)} z_\ell^{m_1^\ell(c)} x_0^{n_0(c)} x_1^{n_1(c)},$$

where

- $m_0^i(c) = |\{j \in T_i \mid c_j = 0\}|$ $(i = 1, \ldots, \ell)$
- $m_1^i(c) = |\{j \in T_i \mid c_j \neq 0\}|$ $(i = 1, \ldots, \ell)$
- $n_0(c) = |\{j \in [n] \setminus \bigcup_{1 \leq i \leq \ell} T_i \mid c_j = 0\}|$
- $n_1(c) = |\{j \in [n] \setminus \bigcup_{1 \leq i \leq \ell} T_i \mid c_j \neq 0\}|$

Jacobi polynomial with respect to ℓ reference vectors

Example (Type III code of length 4 C_{III}^4 (unique))

$$C_{\mathrm{III}}^4 = \{(0,0,0,0),(1,0,1,1),(2,0,2,2),(2,1,0,1),\\ (0,1,1,2),(1,1,2,0),(1,2,0,2),(2,2,1,0),(0,2,2,1)\}$$

If $\mathbf{T} = (\{1,2\},\{4\})$ then

$$\begin{aligned} &J_{C_{\mathrm{III}}^4,(\{1,2\},\{4\})}(w_1,z_1,w_2,z_2,x_0,x_1) \\ &= w_1^2 z_1^0 w_2^1 z_2^0 x_0^1 x_1^0 + \cdots + w_1^1 z_1^1 w_2^0 z_2^1 x_0^0 x_1^1 \\ &= w_1^2 w_2 x_0 + 4 w_1 z_1 z_2 x_1 + 2 z_1^2 w_2 x_1 + 2 z_1^2 z_2 x_0. \end{aligned}$$

Jacobi polynomial with respect to ℓ reference vectors

Example (Type III code of length 4 C^4_{III} (unique))

If $\mathbf{T} = (\{1,2\},\{4\})$ then

$$J_{C^4_{\mathrm{III}},(\{1,2\},\{4\})}(w_1, z_1, w_2, z_2, x_0, x_1)$$
$$= w_1^2 w_2 x_0 + 4w_1 z_1 z_2 x_1 + 2z_1^2 w_2 x_1 + 2z_1^2 z_2 x_0.$$

Then,

$$J_{(C^4_{\mathrm{III}})^\perp,(\{1,2\},\{4\})}(w_1, z_1, w_2, z_2, x_0, x_1)$$
$$= \frac{1}{9} J_{C^4_{\mathrm{III}},(\{1,2\},\{4\})}(w_1 + 2z_1, w_1 - z_1, w_2 + 2z_2, w_2 - z_2, x_0 + 2x_1, x_0 - x_1)$$
$$= \frac{1}{|C^4_{\mathrm{III}}|}(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} \otimes I \otimes I) J_{C^4_{\mathrm{III}},(\{1,2\},\{4\})}(w_1, z_1, w_2, z_2, x_0, x_1).$$

We can obtain a MacWilliams type identity for the Jacobi polynomial of C^4_{III} with respect to 2 reference vectors.

MacWilliams type identity

We can obtain a MacWilliams type identity for the Jacobi polynomial of an $\mathbb{F}_q$-linear code C with respect to ℓ reference vectors as follows:

Theorem (Y. Tanaka et al., 2023)

$$J_{C^\perp,\mathbf{T}}(w_1, z_1, \ldots, w_\ell, z_\ell, x_0, x_1) = \frac{1}{|C|}(\begin{bmatrix} 1 & q-1 \\ 1 & -1 \end{bmatrix} \otimes I \otimes \cdots \otimes I) J_{C,\mathbf{T}}(w_1, z_1, \ldots, w_\ell, z_\ell, x_0, x_1).$$

Theorem (M. Ozeki, 1997)

If C is a binary linear code, then

$$J_{C^\perp,T}(w, z, x, y) = \frac{1}{|C|}(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes I) J_{C,T}(w, z, x, y).$$

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