Generalised Flatness Constants

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Combinatorial and Algebraic Aspects on Lattice Polytopes

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joint with

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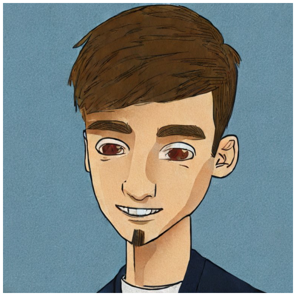

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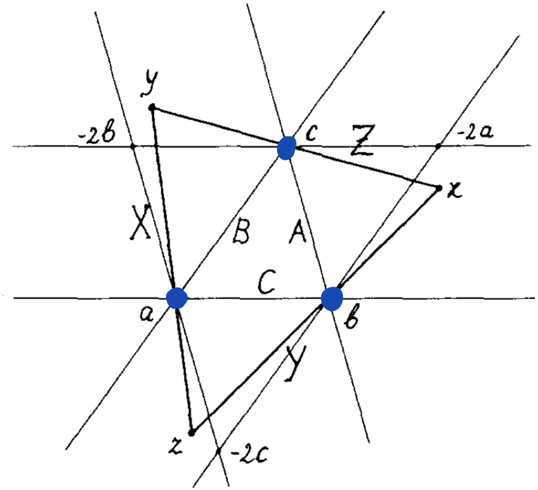
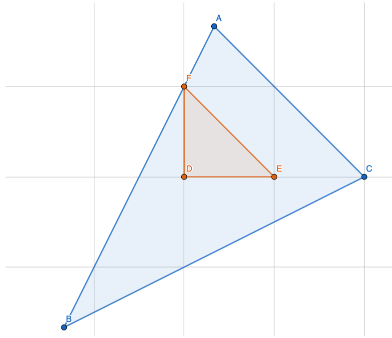
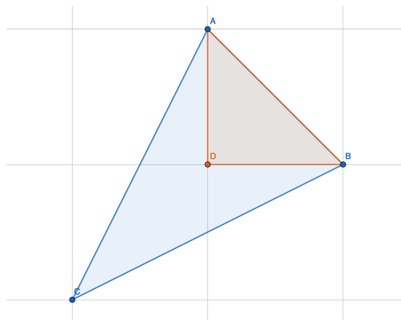
Flt_d



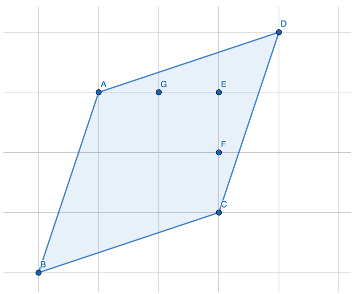
$\mathrm{Flt}_d^{\mathbb{Z}}(S)$



$\mathrm{Flt}_d^{\mathbb{R}}(S)$



$\mathrm{Flt}_d^{\mathrm{span}}$



$K \subseteq \mathbb{R}^d$ convex body

($\text{closed compact convex subset in } \mathbb{R}^d$)

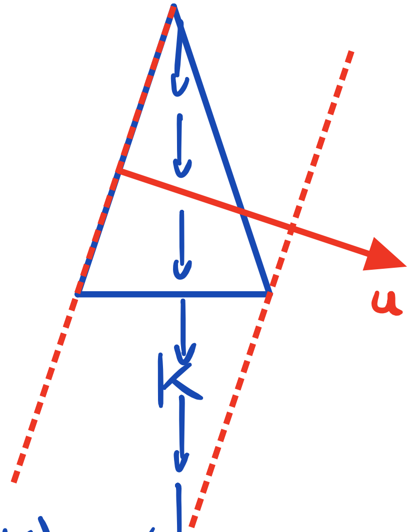
lattice $M = \mathbb{Z}^d$

dual lattice $N = \operatorname{Hom}(M, \mathbb{Z})$

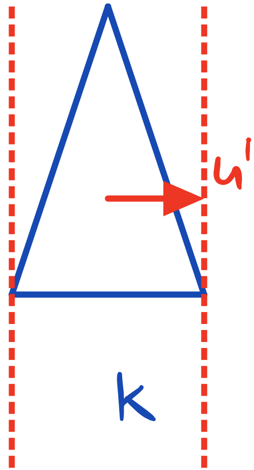
$u \in N$ lattice direction

$$\rightsquigarrow \quad \text{width}_u(K) := \sup_{x,y \in K} |u(x) - u(y)|$$

<u>lattice width</u>: $\text{width}(K) = \min_{u \in N \setminus \{0\}} \text{width}_u(K)$



$\text{width}_u(K) = 6$



$\text{width}(K) = 2$

Classical flatness constant:

$$\mathrm{Flt}_d := \sup\{\mathrm{width}(K) : K \subseteq \mathbb{R}^d \text{ convex body } K \cap \mathbb{Z}^d = \emptyset\}$$

Thm. [Banaszczyk - Litvak - Pajor - Szarek, '99]

$$\mathrm{Flt}_d \leq O(d^{4/3} \log^a(d)) \quad a \text{ const.}$$

Conjecture: Flt_d roughly linear in d

Thm. [Hurkens '90]

$$\mathrm{Flt}_2 = 1 + \frac{2}{\sqrt{3}}$$

Conjecture [Averkov-Codenotti-Macchia-Santos]:

$$\mathrm{Flt}_3 = 2 + \sqrt{2}$$

Thm. [Averkov-Codenotti-Macchia-Santos, '21]:

$$2 + \sqrt{2} \leq \mathrm{Flt}_3 \leq 3.972$$

Generalised flatness constants (Motivation)

lattice polytope $P = \mathrm{conv}(v_1, \ldots, v_n) \subseteq \mathbb{R}^d$

for $v_1, \ldots, v_n \in \mathbb{Z}^d$.

P spanning $:\Longleftrightarrow$ $\mathrm{aff}_{\mathbb{Z}}(P \cap \mathbb{Z}^d) = \mathbb{Z}^d$

Goal: Find easily verifiable certificate for P to be spanning.

Rmk.: **spanning** "mild assumption"

$$\{\text{lattice polytope}\} \longrightarrow \{\text{spanning lattice polytope}\}$$
$$P \subseteq \mathbb{Z}^d \longmapsto P \subseteq \mathrm{aff}_{\mathbb{Z}}(P \cap \mathbb{Z}^d)$$

* **spanning polytopes** have strong properties, e.g.

Thm. (H, Katthän, Nill):

The h^*-vector of spanning polytopes P satisfies for $i+j < \deg(P)$

$$h_1^* + \dots + h_i^* \leq h_{j+1}^* + \dots + h_{j+i}^*$$

(originally known for IDP by Stanley)

unimodular simplex: $S \subseteq \mathbb{R}^d$ simplex
$V(S)$ affine basis of $\mathbb{Z}^d$

Idea: P contains unimodular simplex
$\Rightarrow$ P spanning.

Question: $\mathrm{width}(P) \gg 0 \Rightarrow P$ contains a unimodular simplex?

Thm [ANH, '19]: $K \subseteq \mathbb{R}^d$ convex body
$\mathrm{width}(K) \geq 2\,\mathrm{Flt}_d \cdot d \Rightarrow K$ contains a unimodular simplex.

Think: X is a unimodular simplex

<u>Def.:</u> $X \subseteq \mathbb{R}^d$ bounded set

<u>Flatness constant with respect to X</u>

$$\mathrm{Flt}_d(X) := \sup\{\mathrm{width}(K) : K \subseteq \mathbb{R}^d \text{ convex body, } K \text{ does not contain a unimodular copy of } X\}$$

$Y \in \mathbb{R}^d$ <u>unimodular copy of X</u> if

$\exists\ A \in GL_d(\mathbb{Z})\ \exists\ b \in \mathbb{Z}^d$ s.t.

$$Y = A \cdot X + b.$$

Corollary ~~Thm.~~ [AHN, '19]:

$e_1,\dots,e_d \in \mathbb{Z}^d$ standard basis.

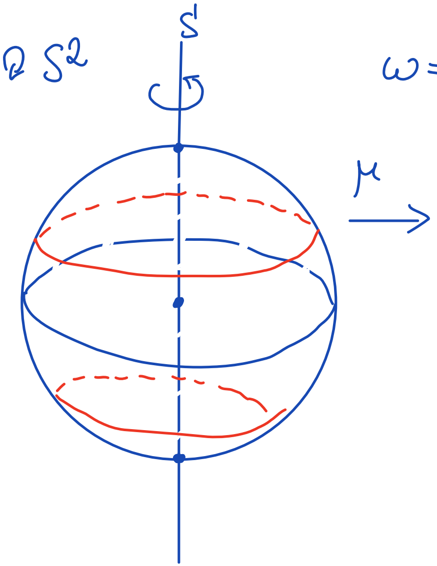
If X fits in a unimodular copy of $n\cdot\Delta_d$ ($\Delta_d = \mathrm{conv}(0, e_1,\dots,e_d)$), then

$$\mathrm{Flt}_d(X) \le 2nd \cdot \mathrm{Flt}_d$$

Question: What is the order of $\mathrm{Flt}_d(n\cdot\Delta_d)$?

Motivation from symplectic geometry:

Example: $S^1 \circlearrowright S^2$



$\omega =$ area form

$\mu(M) =$ interval

$= P$

Def.: Gromov width

$w_G(P) := \sup \{ \underbrace{\pi r^2}_{\text{capacity}} :$ ball of radius r can be symplectically embedded in $M \}$

small adjustment:

$Y \subseteq \mathbb{R}^d$ $\quad$ <u>$\mathbb{R}$-unimodular copy of X</u> $\quad$ if

$$\exists\ A \in GL_d(\mathbb{Z})\ \exists\ b \in \mathbb{R}^d \text{ s.t.}$$

$$Y = A \cdot X + b.$$

<u>Def.:</u> <u>$\mathbb{R}$-Flatness constant with respect to X</u>

$X \subseteq \mathbb{R}^d$ bounded set

$$\mathrm{Flt}_d^{\mathbb{R}}(X) := \sup\{\mathrm{width}(K) : K \subseteq \mathbb{R}^d \text{ convex body, } K \text{ does not contain a } \mathbb{R}\text{-unimodular copy of } X\}$$

Thm. (Latschev, McDuff, Schlenk; Lu; Fang, Littelmann, Pabiniak)

If $R \cdot \Delta_d \subseteq P$, then $\omega_G(P) \geq R$.

Corollary: $\omega_G(P) \geq \dfrac{\mathrm{width}(P)}{\mathrm{Flt}_d \cdot d}$ [AHN, '19]

Ambro-Ito + Larzasfeld

Thm.: ~~Conjecture~~ [AHN, '19]:

$$\omega_G(P) \leq \mathrm{width}(P)$$

$$\frac{\mathrm{width}(P)}{\mathrm{Flt}_d \cdot d} \leq \omega_G(P) \leq \mathrm{width}(P)$$

Further connections to Seshadri constants in symplectic / algebraic geometry:

$$\mathrm{Flt}_d^{\mathrm{span}} := \sup\{\mathrm{width}(K) : K \subseteq \mathbb{R}^d \text{ convex body}, \; \mathrm{int}(K) \cap \mathbb{Z}^d \text{ does not span } \mathbb{Z}^d\}.$$

$$\mathrm{Flt}_d^{\mathrm{basis}} := \sup\{\mathrm{width}(K) : K \subseteq \mathbb{R}^d \text{ convex body}, \; \mathrm{int}(K) \text{ does not contain a basis of } \mathbb{Z}^d\}$$

$\mathrm{Flt}_d(\Delta_2)$

Thm. [Ambro–Ito, 2020]

$K \subseteq \mathbb{R}^d$ convex body, $\dim(K) = d$

lattice width w

(i) $w > d^2 + d \Rightarrow \operatorname{int}(K)$ contains a unimodular simplex

(ii) $w > 2d^2 \Rightarrow M \cap \operatorname{int}(K)$ spans $\mathbb{Z}^d$

(iii) $\sqrt[d]{d! \cdot |\mathbb{Z}^d \cap \operatorname{int}(K)|} \geq \frac{w}{d} - d$

.....

2.2-dim M

$$\frac{\mathrm{width}(P)}{2} \le w_G(M) \le \mathrm{width}(P)$$

Explicit computations:

Thm [CHH, '21]: $\mathrm{Flt}_2^{\mathbb{R}}(\Delta_2) = 2$

$\mathrm{Flt}_2^{\mathbb{Z}}(\Delta_2) = \frac{10}{3}$

idea: $K \subseteq \mathbb{R}^d$ convex set $\underline{\mathbb{R}/\mathbb{Z}\text{-}X\text{-free}}$ (A)

if $\mathrm{relint}(K)$ contains no $\mathbb{R}/\mathbb{Z}$-unimodular copy of X (A)

Thm (CHH, '21):

$A = \mathbb{Z}$ ☹

$A = \mathbb{R}$ ✓

$X \subseteq \mathbb{R}^d$ full-dim. polytope

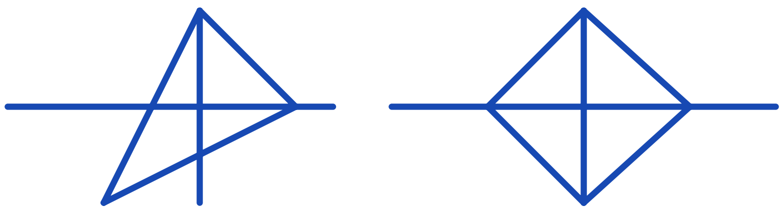
$\Rightarrow$ each inclusion-maximal $\overset{A}{\mathbb{R}/\mathbb{Z}}$-$X$-free convex body $K \subseteq \mathbb{R}^d$ is a polytope.

$$\sup \left\{ \mathrm{width}(K) \;\middle|\; \begin{array}{l} K \text{ incl.-max.} \\ \mathbb{R}/\mathbb{Z}\text{-}X\text{-free} \\ \text{convex body} \end{array} \right\} \leq \mathrm{Flt}_d^{\mathbb{R}/\mathbb{Z}}(X) \leq \sup \left\{ \mathrm{width}(K) \;\middle|\; \begin{array}{l} K \text{ incl.-max.} \\ \mathbb{R}/\mathbb{Z}\text{-}X\text{-free} \\ \text{convex set} \end{array} \right\}$$

<u>Corollary:</u> * There is unique $\mathbb{Z}$-Δ_2-free inclusion-maximal polygon:

$$\operatorname{conv}\left(\begin{bmatrix} 1/3 \\ 5/3 \end{bmatrix}, \begin{bmatrix} -4/3 \\ -5/3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right)$$

* There are infinitely many $\mathbb{R}$-Δ_2-free inclusion-maximal polygons, e.g.



Particularly interesting case for symplectic geometry:

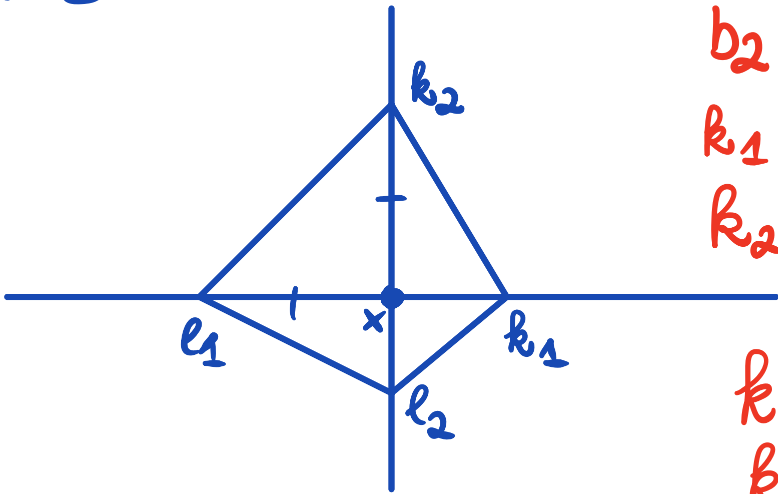
linearly independent   center   size

$b_1, \dots, b_d \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, $a \in \mathbb{Z}_{\geq 1}$, $k_1, \dots, k_d$,
$\ell_1, \dots, \ell_d \in \mathbb{R}_{\geq 0}$ s.t. $k_i + \ell_i = a$

$$D := x + \operatorname{conv}(k_1 b_1, -\ell_1 b_1, \dots, k_d b_d, -\ell_d b_d)$$

diamond of size a

Example: $a = 3$



$b_1 = e_1$

$b_2 = e_2$

$k_1 = \ell_2 = 1$

$k_2 = \ell_1 = 2$

$k_1 + \ell_1 = 3$

$k_2 + \ell_2 = 3$

Question: What is $\mathrm{Flt}_a^{\mathbb{R}}$(diamond)?

Thank You

For Your Attention