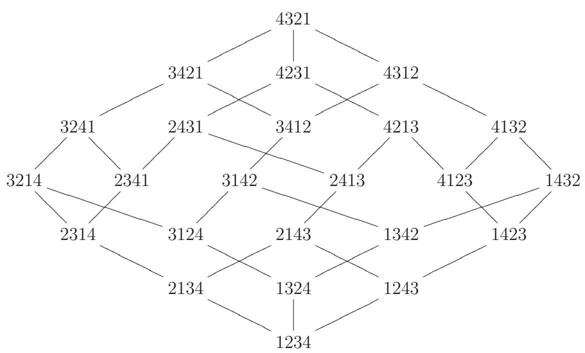
COMBINATORICS ON BIGRASSMANNIAN PERMUTATIONS AND ESSENTIAL SETS

MASATO KOBAYASHI

Contents



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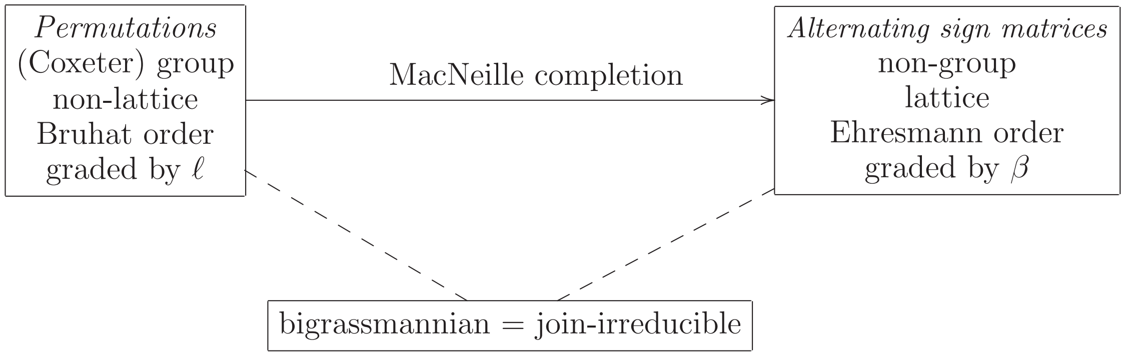


FIGURE 1. Structures of two classes of matrices

1. SYMMETRIC GROUPS

Introduction. Symmetric groups

Permutation statistics:

Coxeter length major index

MacMahon:

$$\sum_{w \in S_n} q^{\ell(w)} =$$

Mahonian, Eulerian, .

S_n as a Coxeter group.

Notation.

$W =$ $S =$

$T =$ $\ell =$

Inversion set

$I(w) =$

Def (height, weak order, Bruhat order)**.**

Bigrassmannian permutations?

Def (bigrassmanninan permutations)**.**

[Four-box construction]
$\boxed{123\cdots n} \to \boxed{}\boxed{}\boxed{}\boxed{}$.

$\boxed{1}\boxed{34567}\boxed{2}\boxed{8}$, $\boxed{}\boxed{}\boxed{}\boxed{}$, $\boxed{}\boxed{}\boxed{}\boxed{}$ and $\boxed{}\boxed{}\boxed{}\boxed{}$.

Question.

Def (join-irreducibility)**.**

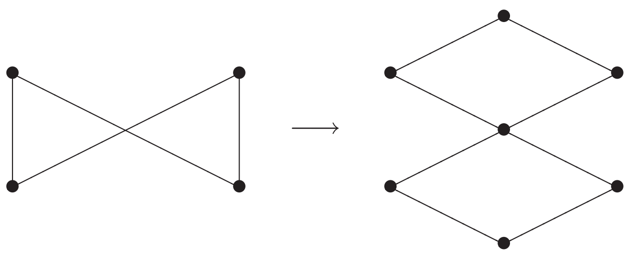
Fact (MacNeille completion)**.**

Fact (Lascoux-Schützenberger 1996)**.** The following are equivalent:

(1)

(2)

Figure 2. MacNeille completion



Bigrassmannian statistics.

Def (bigrassmannian statistic).

$$\beta(w) =$$

Fact (Kobayashi 2011).

$$\beta(w) = \sum_{\substack{i<j \\ w(i)>w(j)}} (w(i) - w(j)) = \frac{1}{2}\sum_{i=1}^{n}(i - w(i))^2.$$

Example.

$$\beta(3412) =$$

$$\beta(4321) =$$

Def (signed bigrassmannian polynomials).

$$B_n(q) =$$

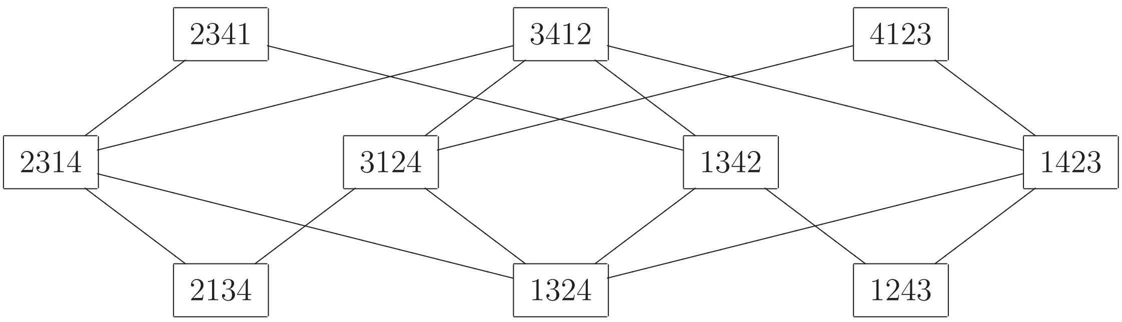


FIGURE 3. Bruhat order of bigrassmannian permutations

Signed bigrassmannian statistics.

Def (q-analog of a matrix).

Proposition (determinantal expression).

Example.

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}_q = \det \begin{pmatrix} 1 & q^{1/2} & q^{4/2} & q^{9/2} \\ q^{1/2} & 1 & q^{1/2} & q^{4/2} \\ q^{4/2} & q^{1/2} & 1 & q^{1/2} \\ q^{9/2} & q^{4/2} & q^{1/2} & 1 \end{pmatrix}$$

$$=$$

(Open problem)

-
-
-

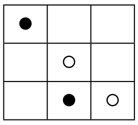
Table 1. signed bigrassmannian statistic over S_4

	sign	β		sign	β		sign	β		sign	β
1234	+	0	2134	−	1	3124	+	3	4123	−	6
1243	−	1	2143	+	2	3142	−	5	4132	+	7
1324	−	1	2314	+	3	3214	−	4	4213	+	7
1342	+	3	2341	−	6	3241	+	7	4231	−	9
1423	+	3	2413	−	5	3412	+	8	4312	−	9
1432	−	4	2431	+	7	3421	−	9	4321	+	10

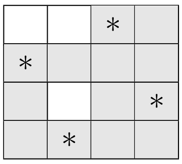
2. Essential sets

Essential sets and Baxter permutations.

Def (colored diagram)**.**



Def (Rothe diagram, essential set)**.**



Def (Baxter permutations)**.**

Fact (Eriksson-Linusson)**.** For $x \in S_n$, the following are equivalent:

(1) x is Baxter.

(2)

Corollary. *bigrassmannian* $\Longrightarrow$ *Baxter.*

dual essential sets.

Def (dual essential set)**.**

Def (essential diagram)**.**

Theorem (Kobayashi 2013)**.** *The following are equivalent:*

(1) *x is Baxter.*

(2)

Theorem (cluster-like structure, Kobayashi 2013)**.** *If $y = xs_i$, $y(i) > y(i+1)$, then*

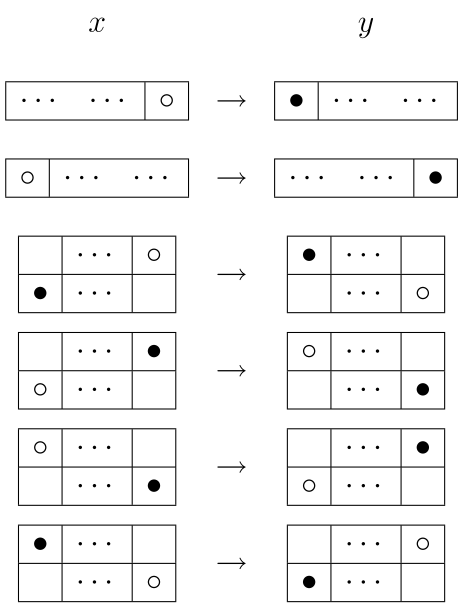
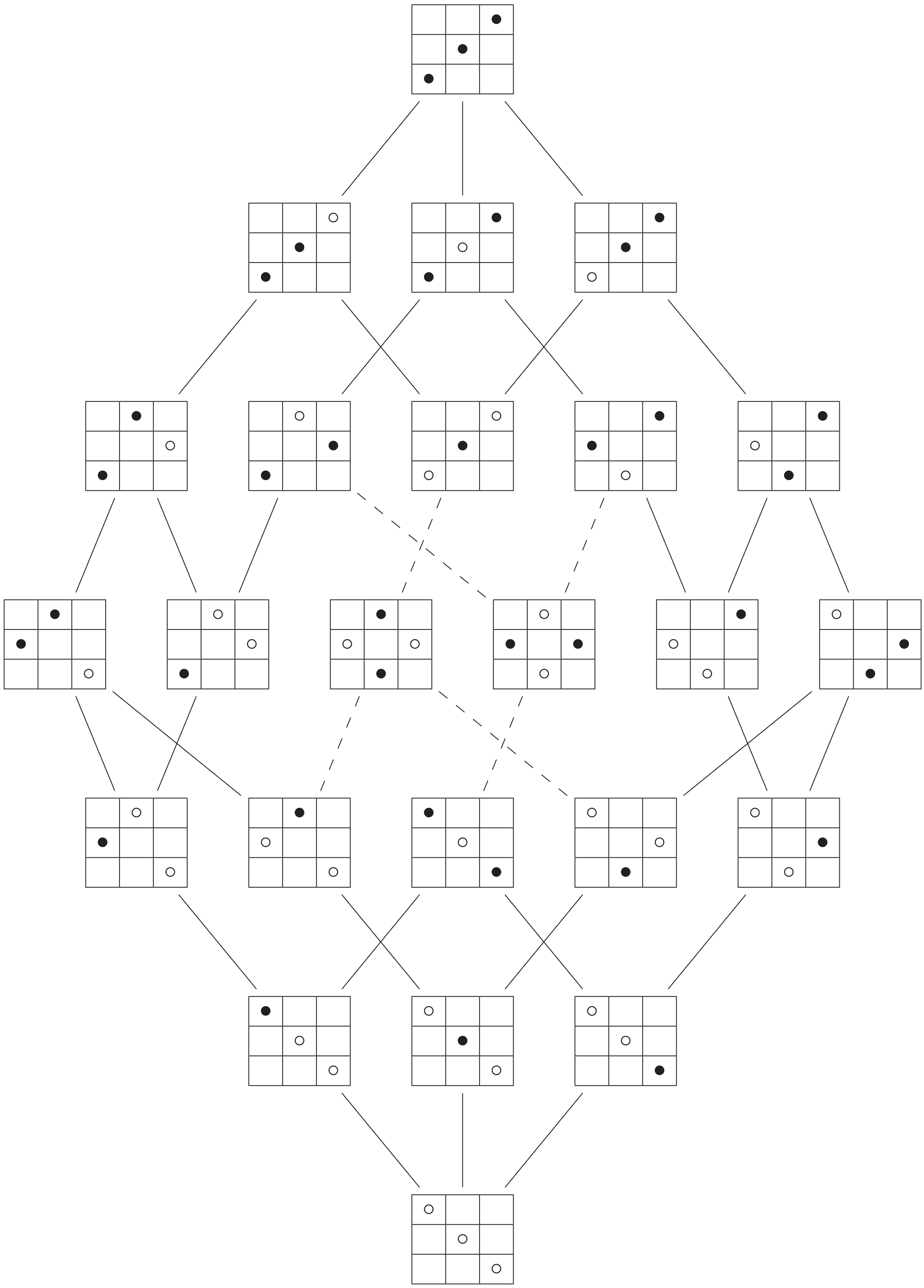


Figure 4. Essential diagrams on S_4



ranked diagrams.

Def (ranked diagram)**.**

$\widetilde{x}(i,j) =$

$$\widetilde{231} = \begin{array}{|c|c|c|}\hline 0 & 1 & 1 \\ \hline 0 & 1 & 2 \\ \hline 1 & 2 & 3 \\ \hline \end{array}.$$

Proposition. *The following are equivalent:*

(1) $x \le y$
(2) $\widetilde{x}(i,j) \ge \widetilde{y}(i,j)$ *for all* i, j.

Proposition (Essential conditions)**.** *Let* $x \in S_n$ *and* $(i,j) \in [n-1]^2$. *Then*

$$\begin{array}{lll} (1) & j < x(i) & \iff \widetilde{x}(i-1,j) = \widetilde{x}(i,j). \\ (2) & i < x^{-1}(j) & \iff \widetilde{x}(i,j-1) = \widetilde{x}(i,j). \\ (3) & x(i+1) \le j & \iff \widetilde{x}(i+1,j) = \widetilde{x}(i,j)+1. \\ (4) & x^{-1}(j+1) \le i & \iff \widetilde{x}(i,j+1) = \widetilde{x}(i,j)+1. \end{array}$$

Theorem (Kobayashi)**.** *The following are equivalent:*

(1) $x \le y$
(2) $\widetilde{x}(i,j) \ge \widetilde{y}(i,j)$ *for all* $i, j \in \mathrm{Ess}(x)$.

Def (Fulton diagram)**.**

Example. $\widetilde{13254} = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 2 & 2 \\ \hline 1 & 2 & 3 & 3 \\ \hline 1 & 2 & 3 & 3 \\ \hline \end{array}$ and $\widetilde{35241} = \begin{array}{|c|c|c|c|}\hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 1 & 2 & 2 \\ \hline 0 & 1 & 2 & 3 \\ \hline \end{array}$.

$$F(13254) = \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & 3 \\ \hline \end{array} \text{ and } \widetilde{35241}|_{\mathrm{Ess}(13254)} = \begin{array}{|c|c|c|c|}\hline 0 & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & 3 \\ \hline \end{array}$$

Two interpretations.
Coxeter group method:

Proposition (Deodohar).

Finite distributive lattice method:

Proposition (Lascoux-Schützenberger).

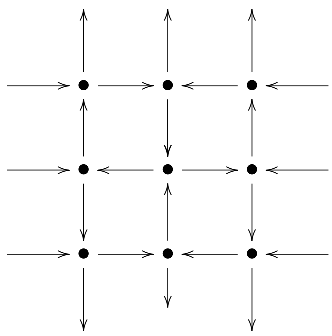
3. Alternating sign matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

alternating sign matrices

2		
1	3	
1	2	3

monotone triangles



6 vertex model

Def (Alternating sign matrices).

Hisotry: ASM conjecture. Robbins-Rumsey (1983)

$$|A_n| = \qquad\qquad \text{as} \qquad \begin{matrix} 1 \\ 2 \\ 7 \\ 42 \\ 7436 \\ 218348 \\ 10850216 \\ 911835460 \\ 129534272700 \\ 31095744852375 \\ 12611311859677500 \\ \vdots \end{matrix}$$

$$\begin{cases} \text{Zeilberger (1995)} \\ \text{Kuperberg (1996)} \end{cases}$$

Byproducts.

Def (Corner sum matrices, essential sets)**.**

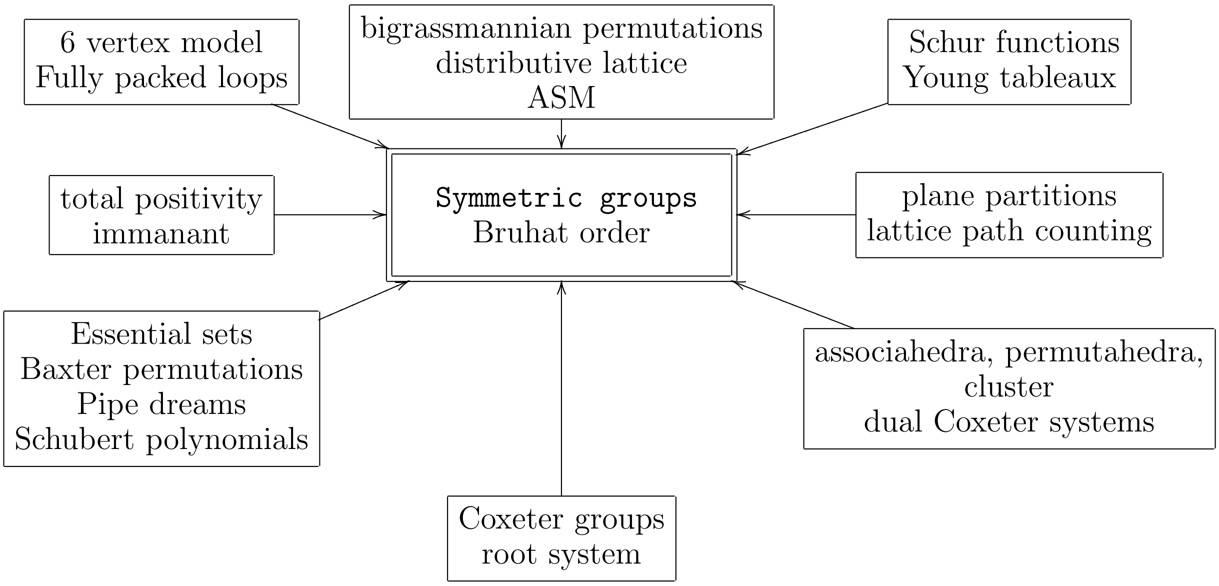
Def (bigrassmannian statistics)**.**

Corollary (essnetial criterion for ASMs)**.** *The following are equivalent:*

(1)

(2)

Proposition (cluster-like structure, Kobayashi)**.**



Open problems.

- ASM polytope
- Cluster structure
- type BC, D, affine
- unsigned bigrassmannian statistic

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